An approximate Tutte-decomposition for arbitrary connectivity

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with Romain Bourneuf, Johannes Carmesin and Tim Planken

SIAM DM '26

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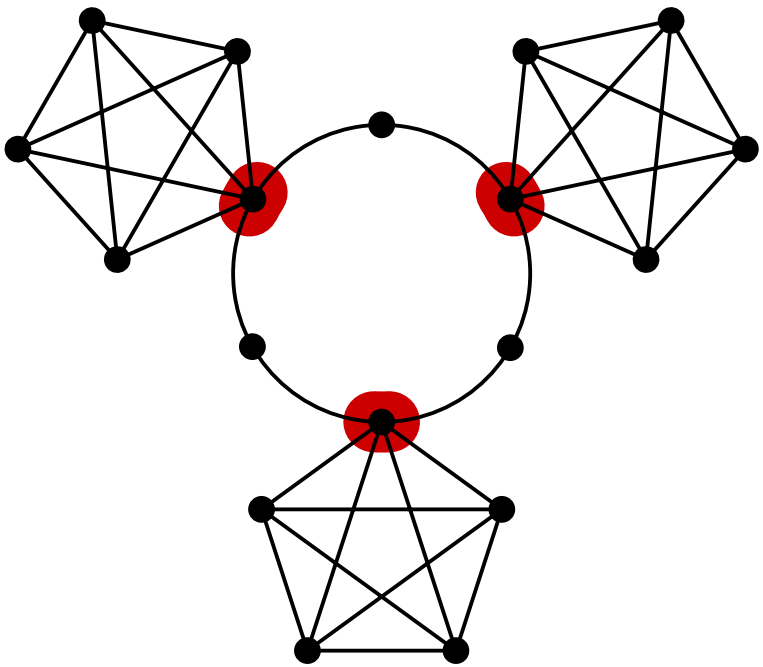
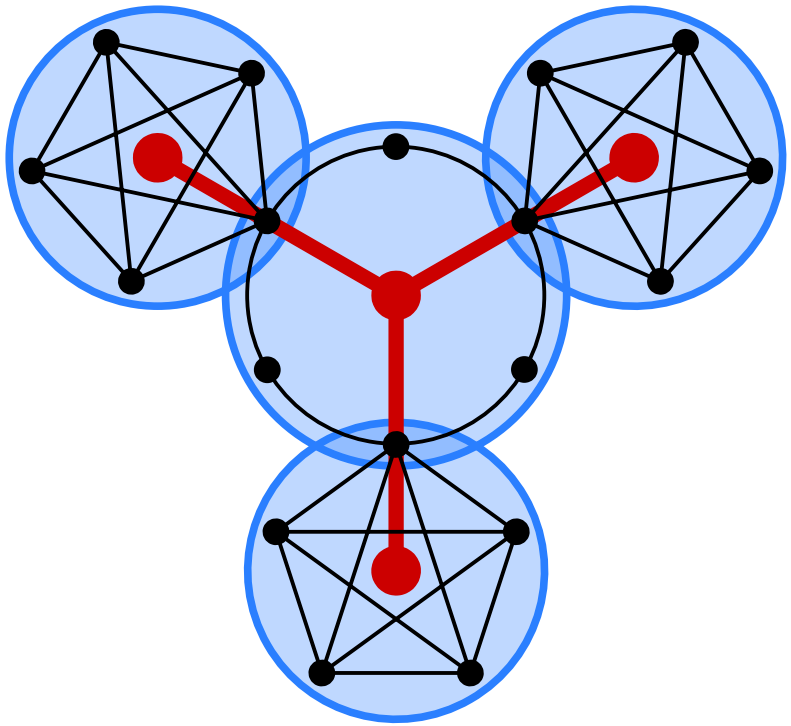
Problem: Every k-con'd G has a tree-decomposition along k-separators into torsos that are $(k+1)$-con'd or 'basic'.

$k = 1$:





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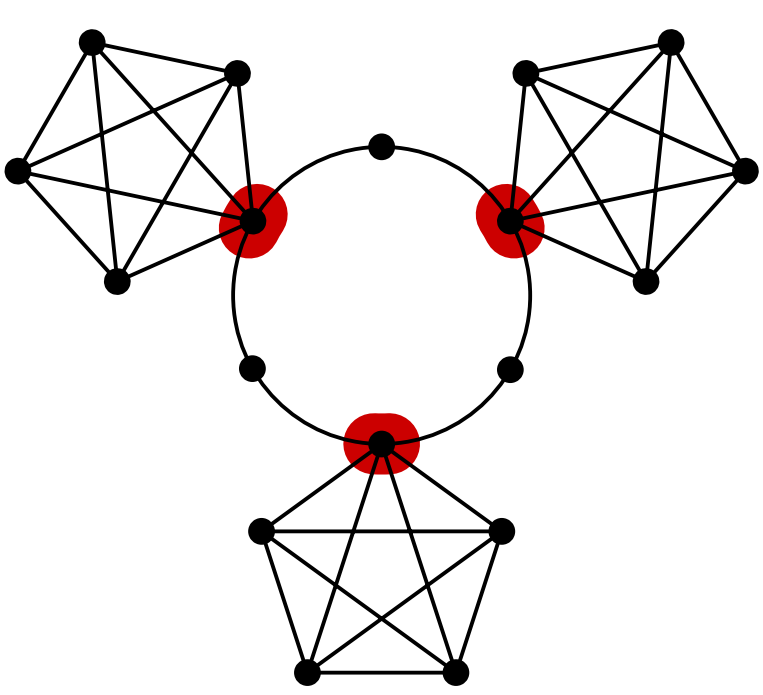

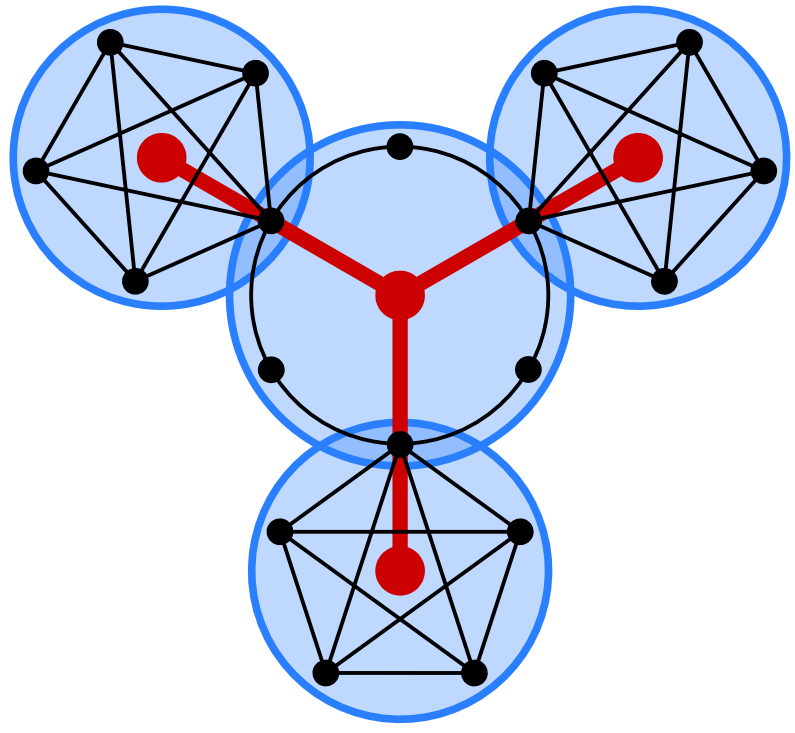

$(T, (V_t)_{t \in V(T)})$ where T is *decomposition-tree*, $V_t \subseteq V(G)$ are *bags*,

- $G = \bigcup \{ G[V_t] : t \in T \}$
- $\forall v \in V(G) : \quad \{ t \in T : v \in V_t \}$ is con'd in T.

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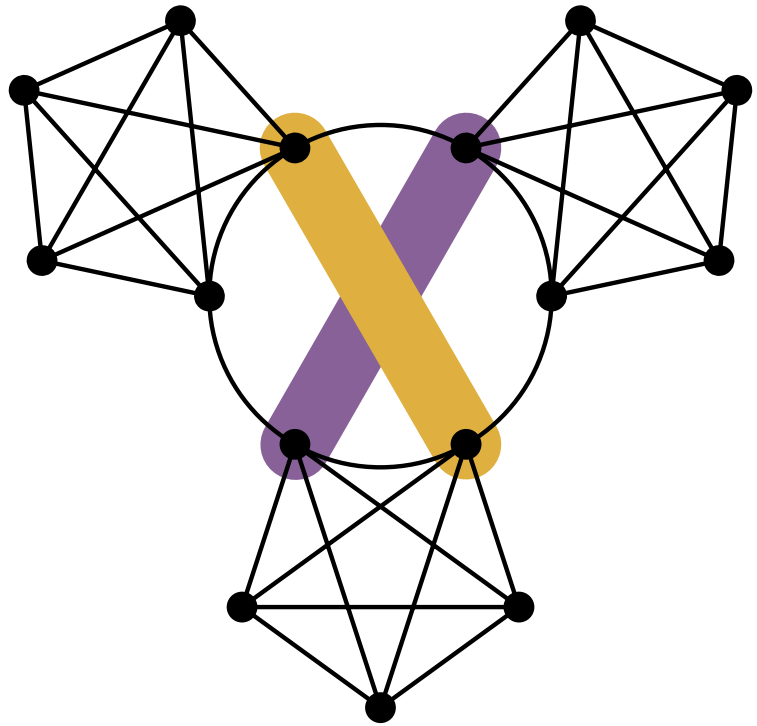

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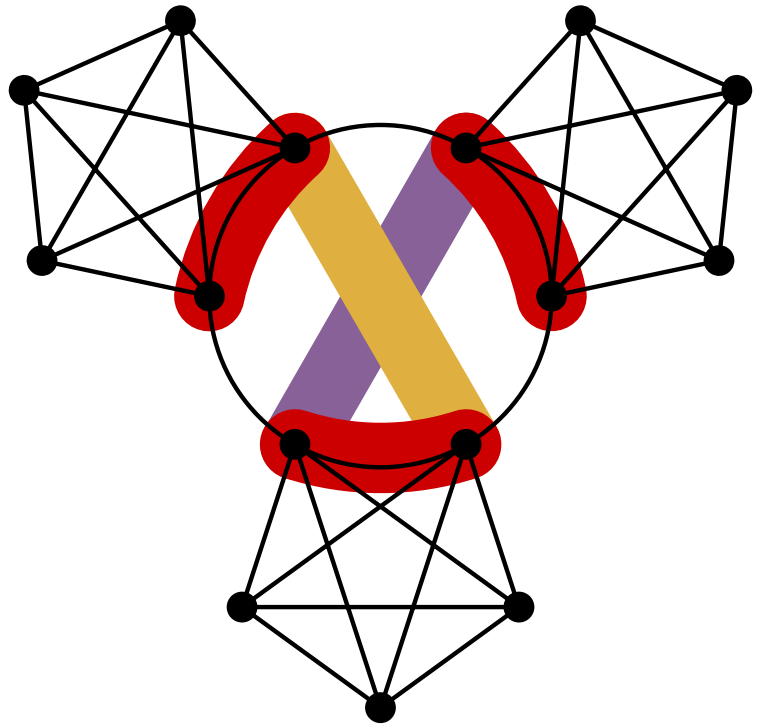

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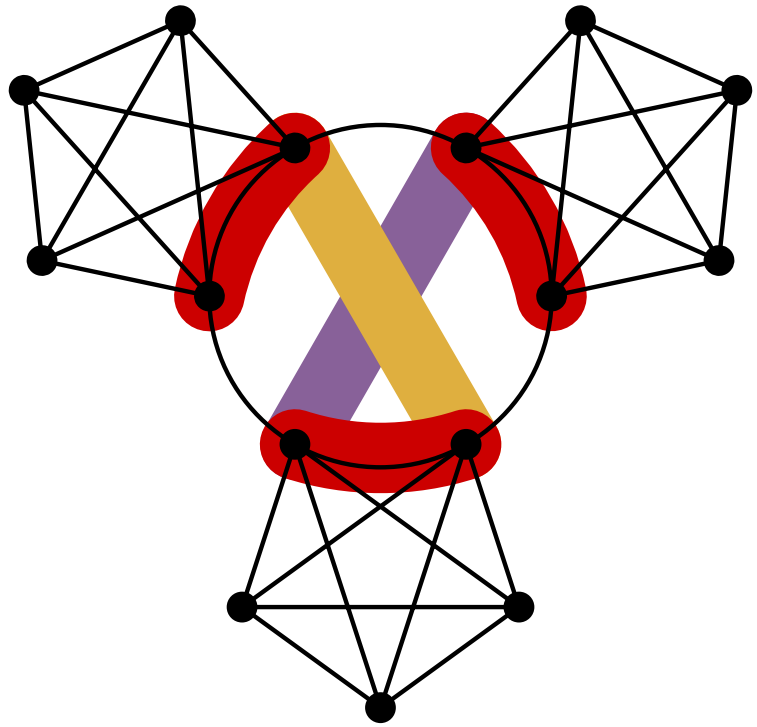

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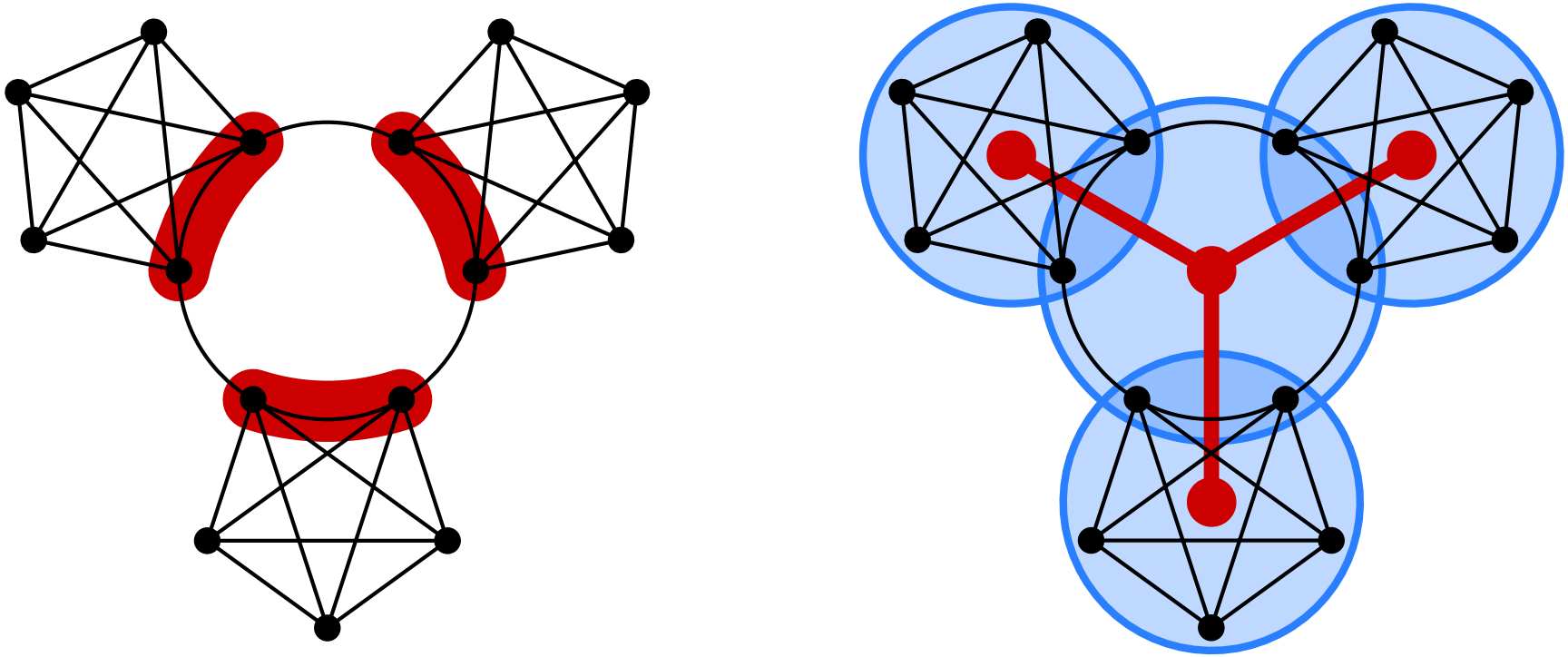

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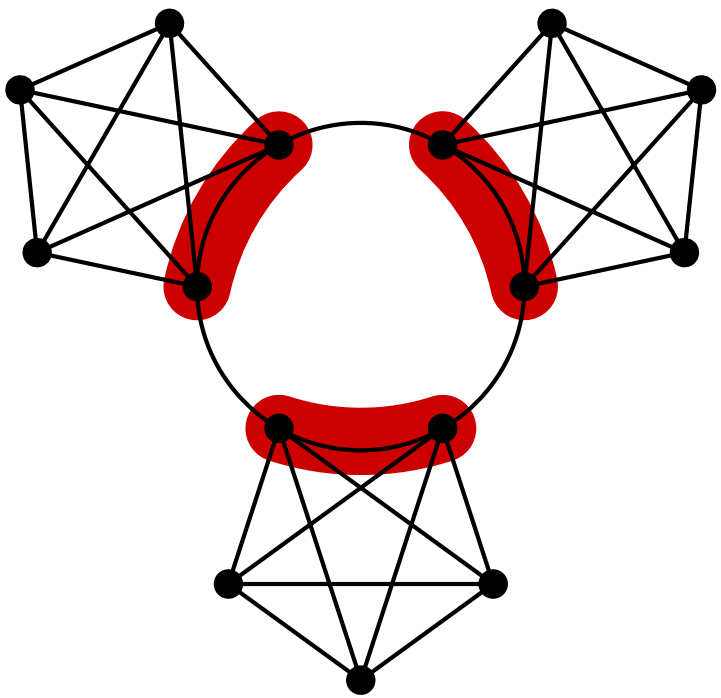

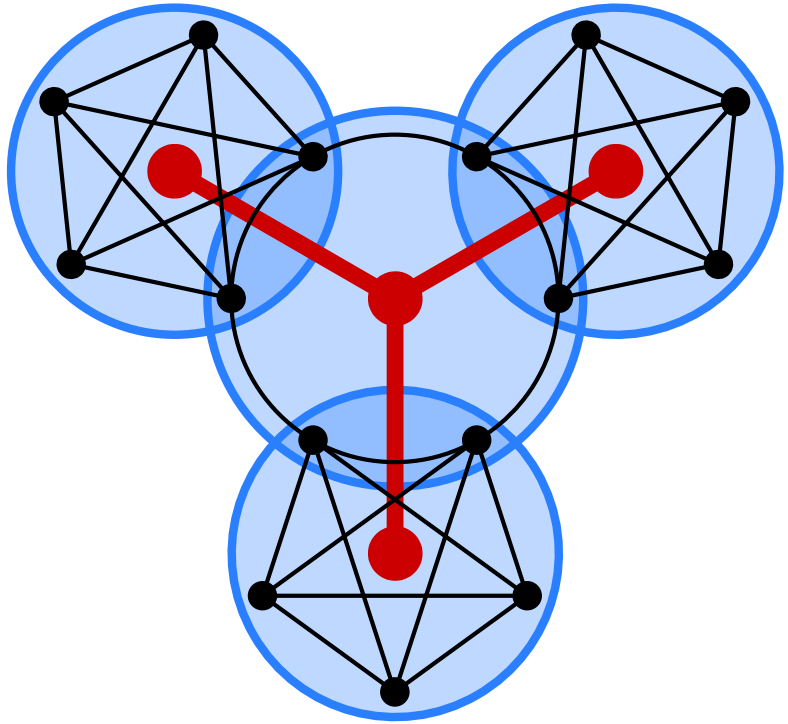

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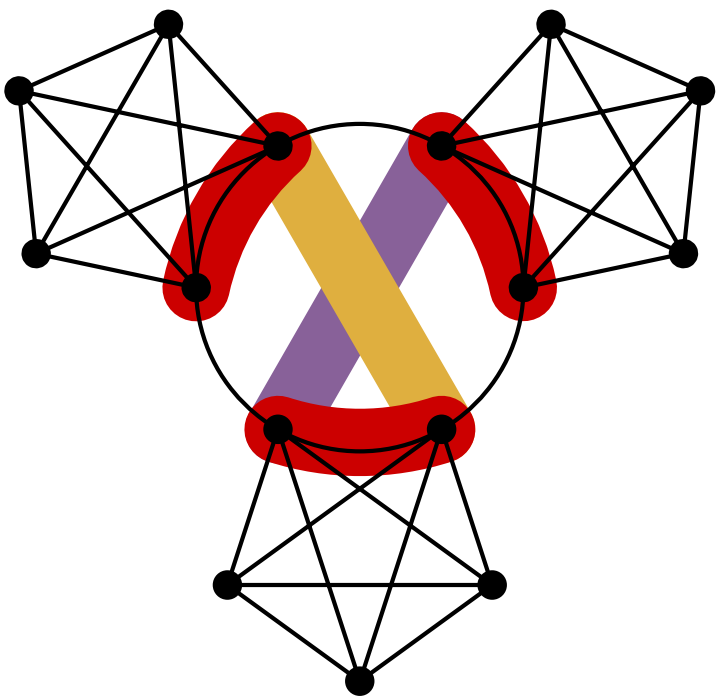

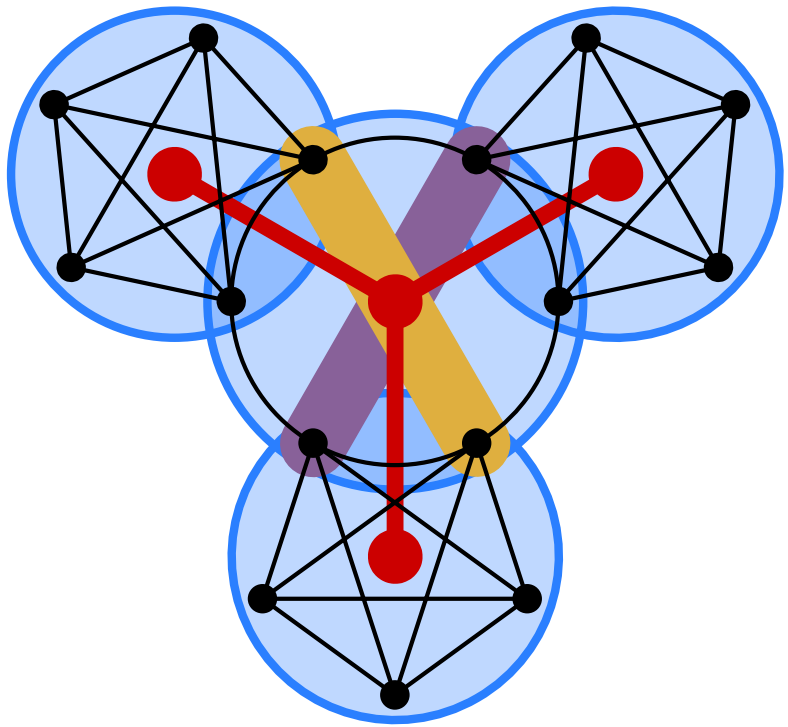

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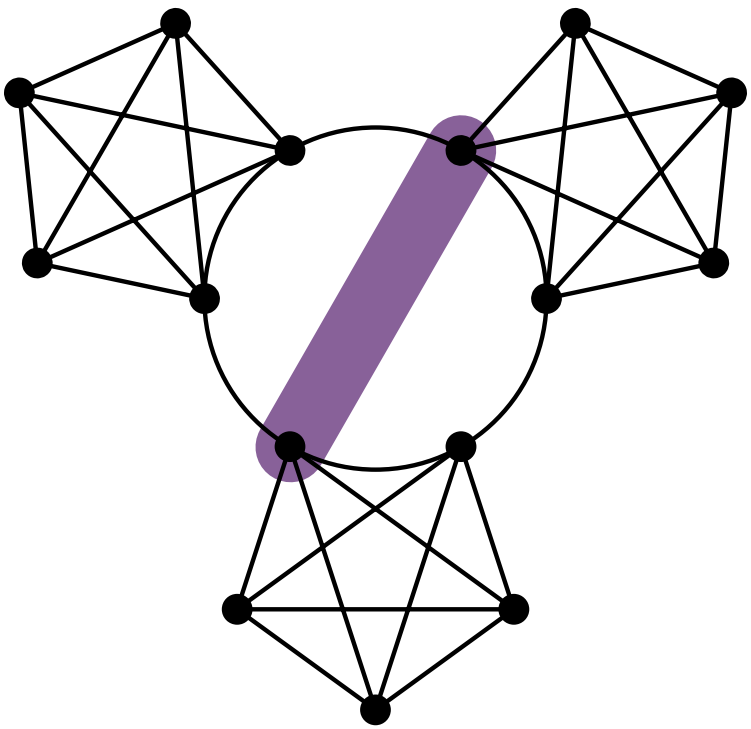

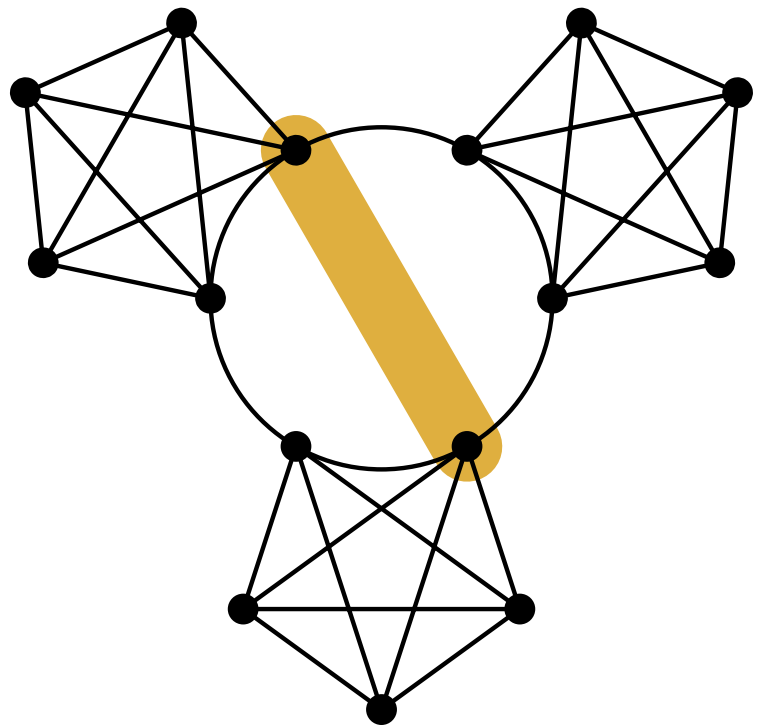

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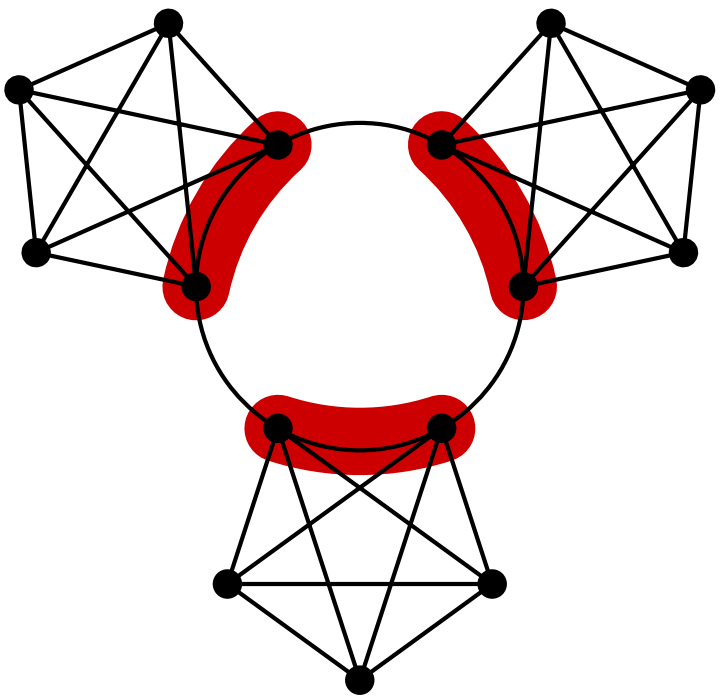

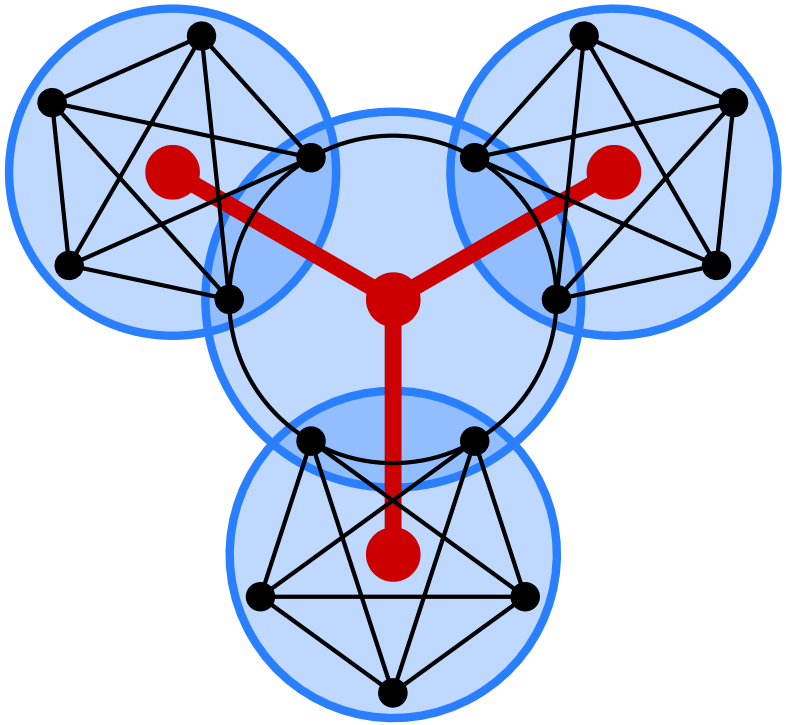

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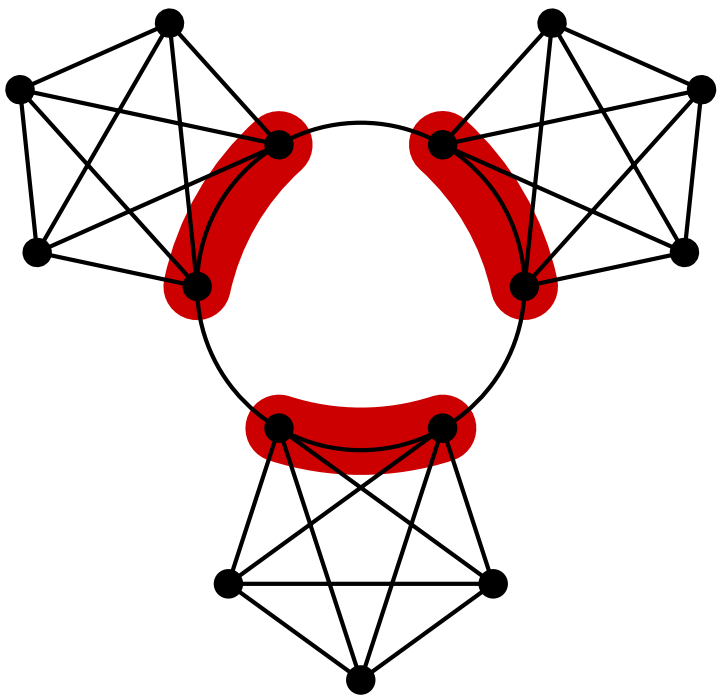

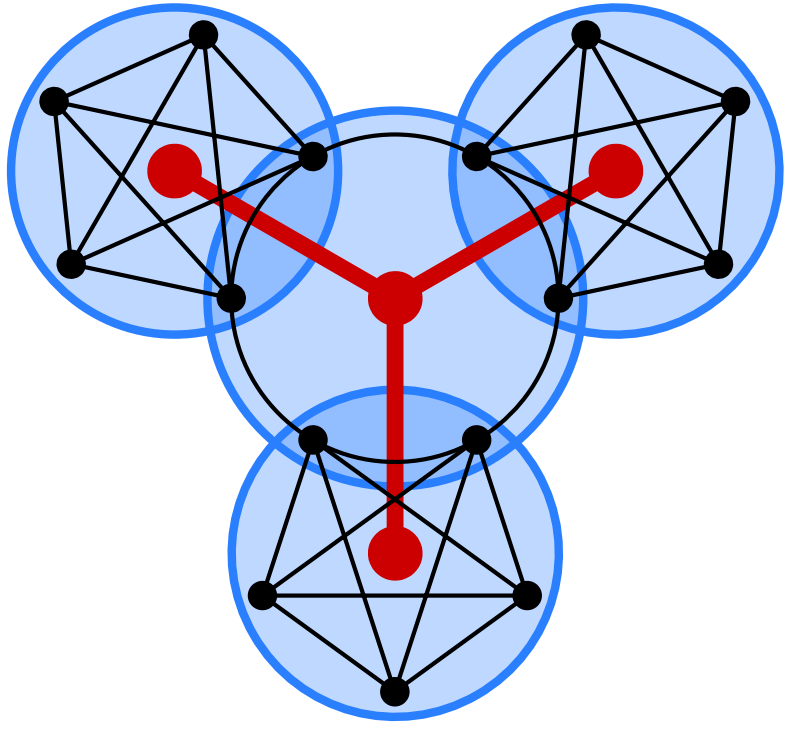

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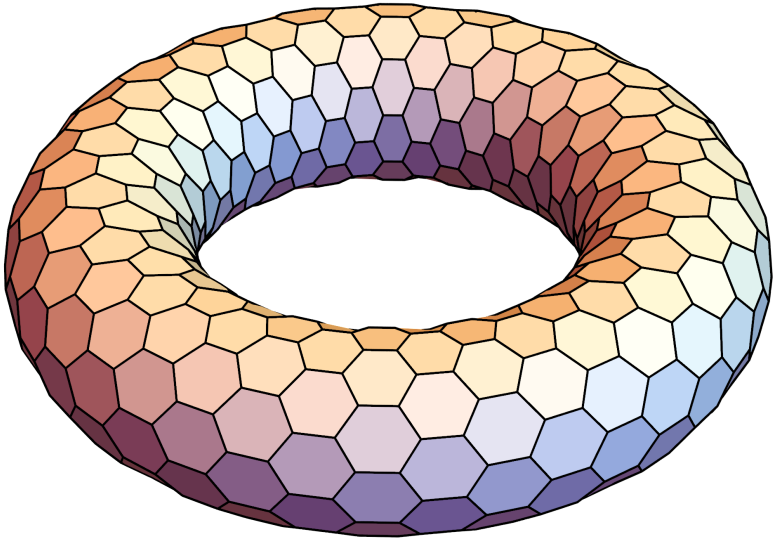

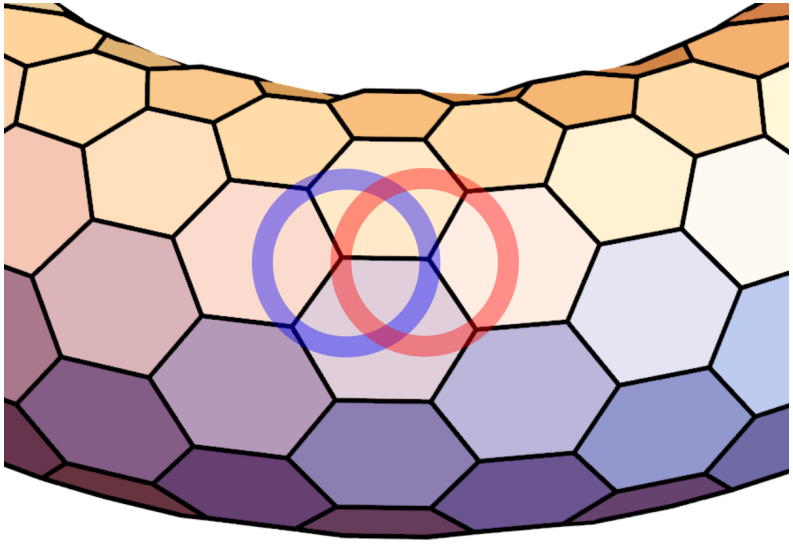

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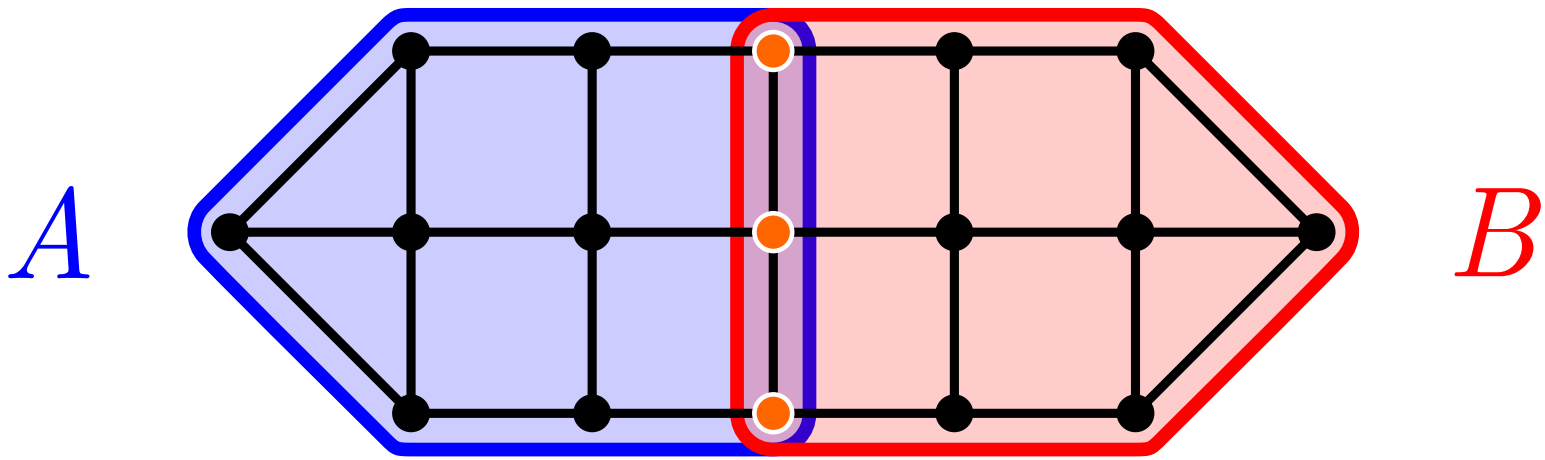
G is *q-almost-k-connected* $:\Longleftrightarrow$ every $(< k)$-sep'n has breadth $\leqslant q$.



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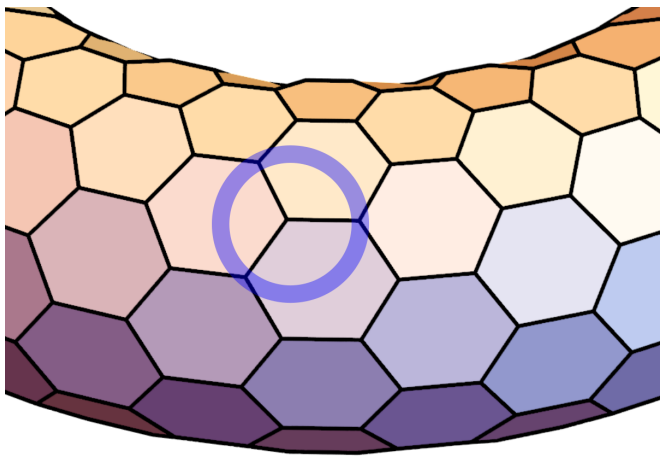

1-almost-4-con'd

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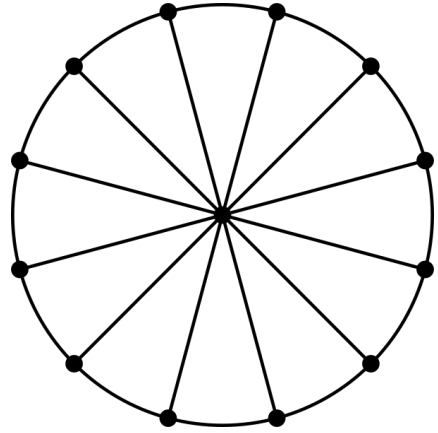
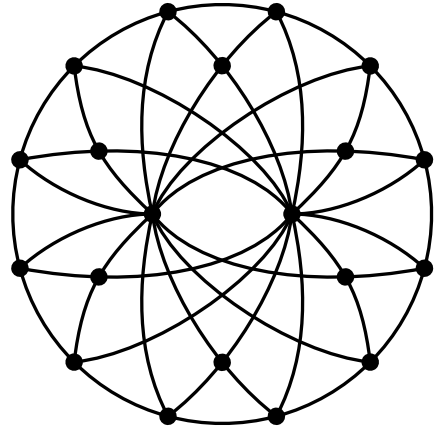
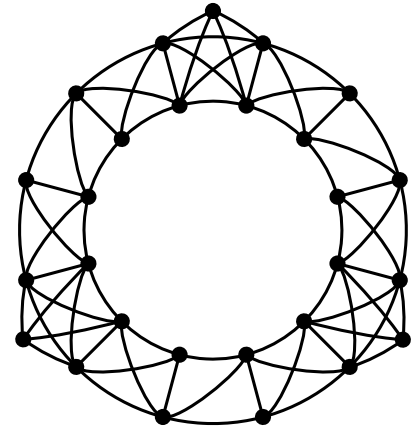
3-connectivity:	(Carmesin & K. ’23) 
4-connectivity:	(K. & Planken ’25)  

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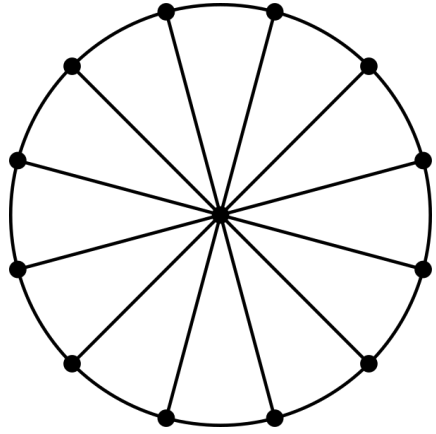
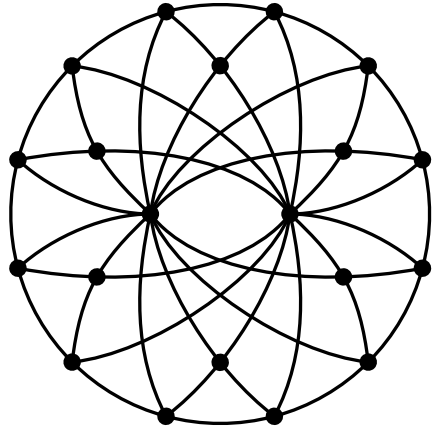
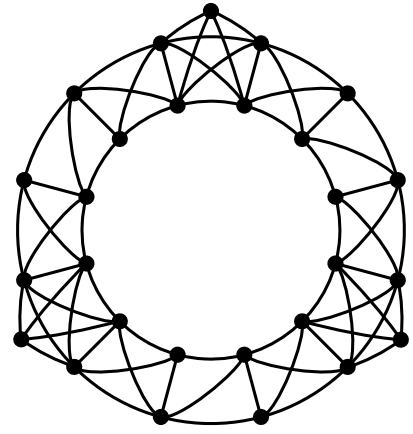
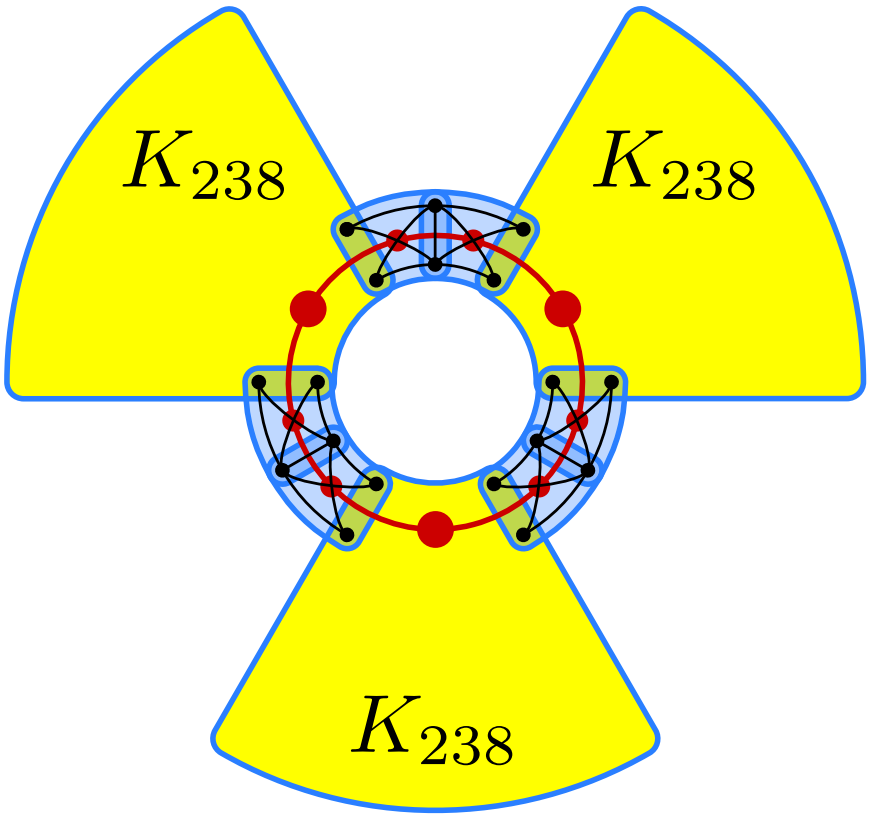
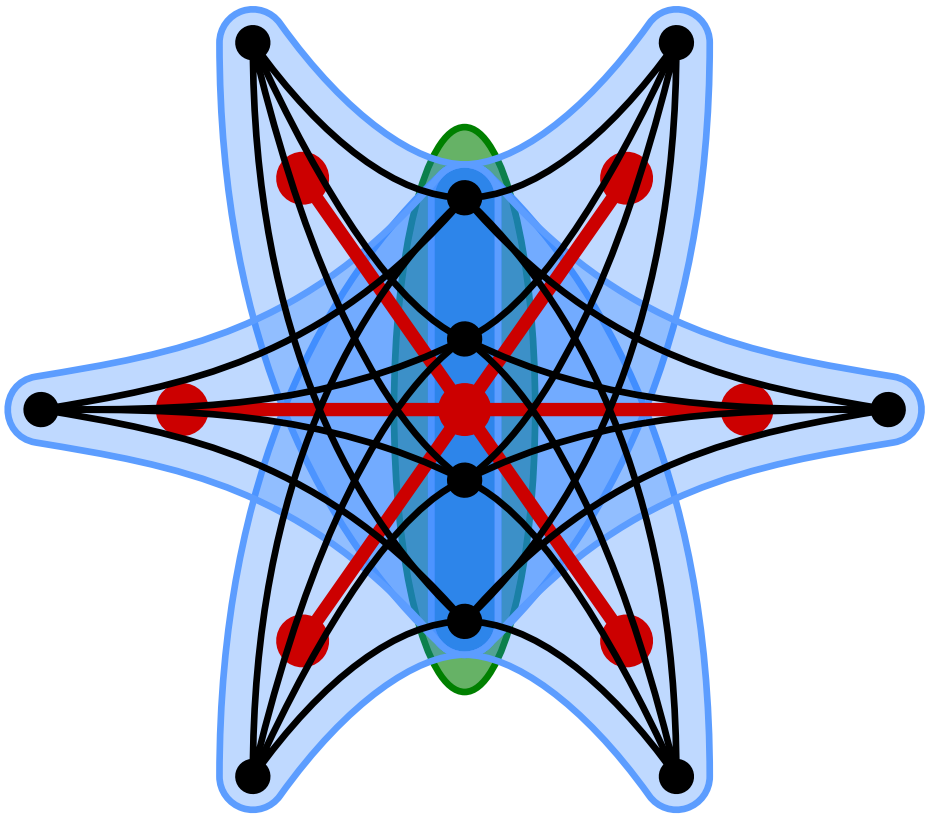
3-connectivity:	(Carmesin & K. ’23) 
4-connectivity:	(K. & Planken ’25)  

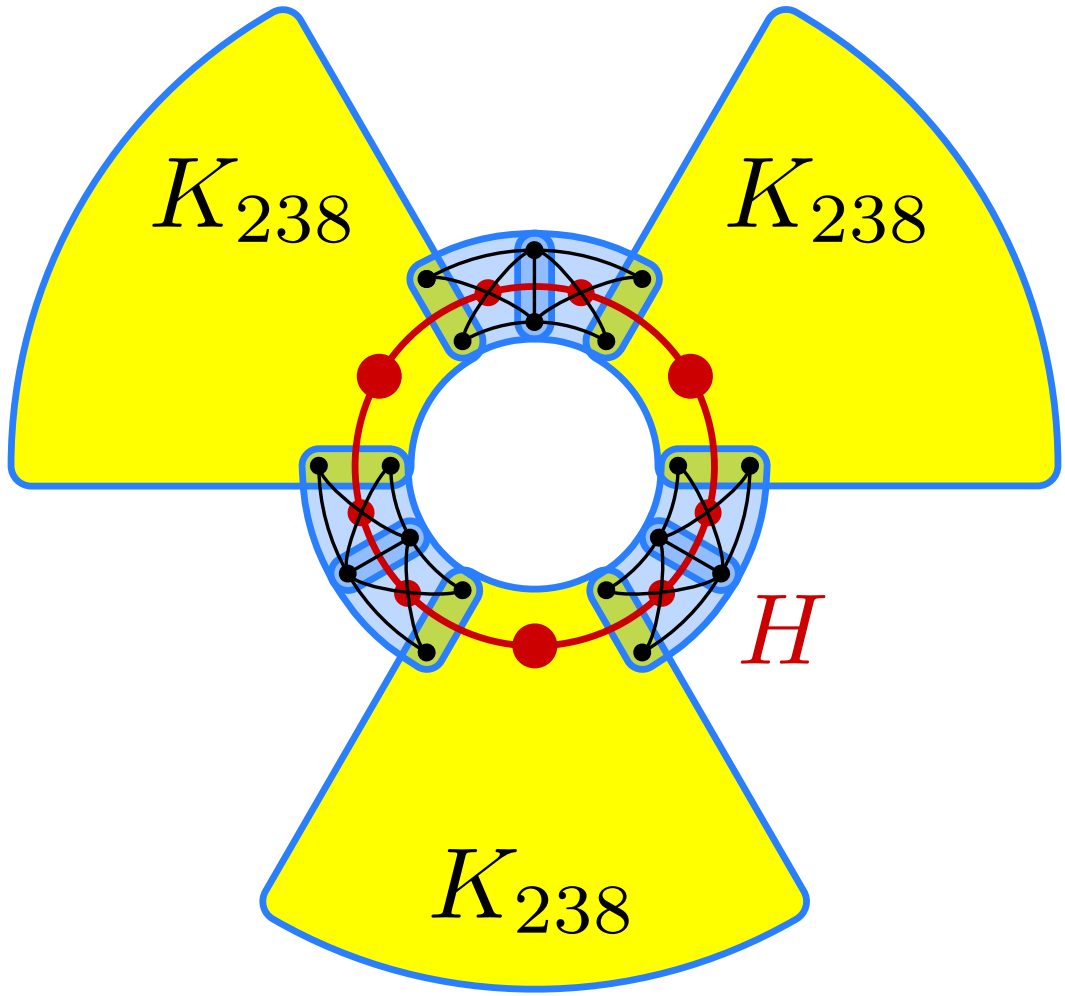
$\Longrightarrow$ Idea: use flowers by Oxley, Semple and Whittle!

A *k-flower* is a k-daisy or a k-anemone.





Let $(H, \mathcal{V})$ be a **cycle**-decomposition of G.



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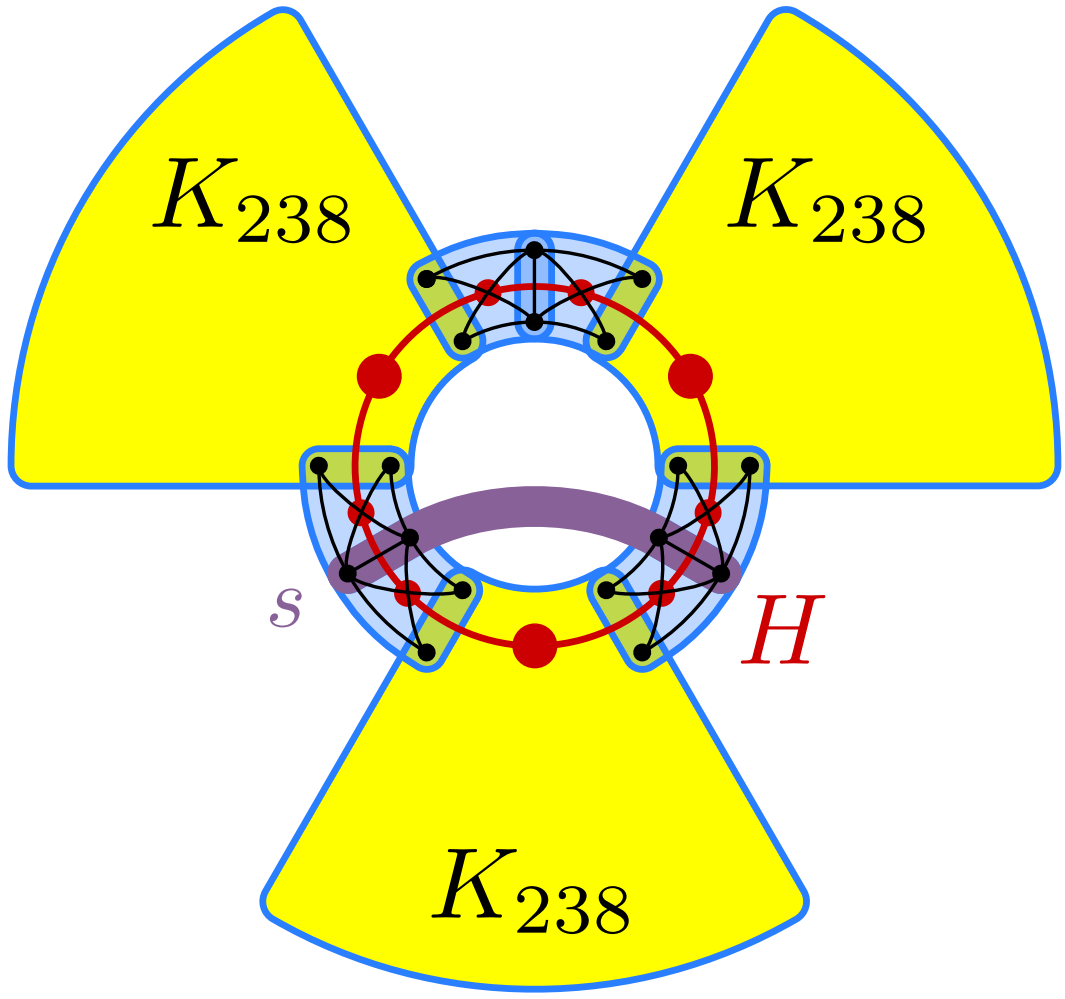

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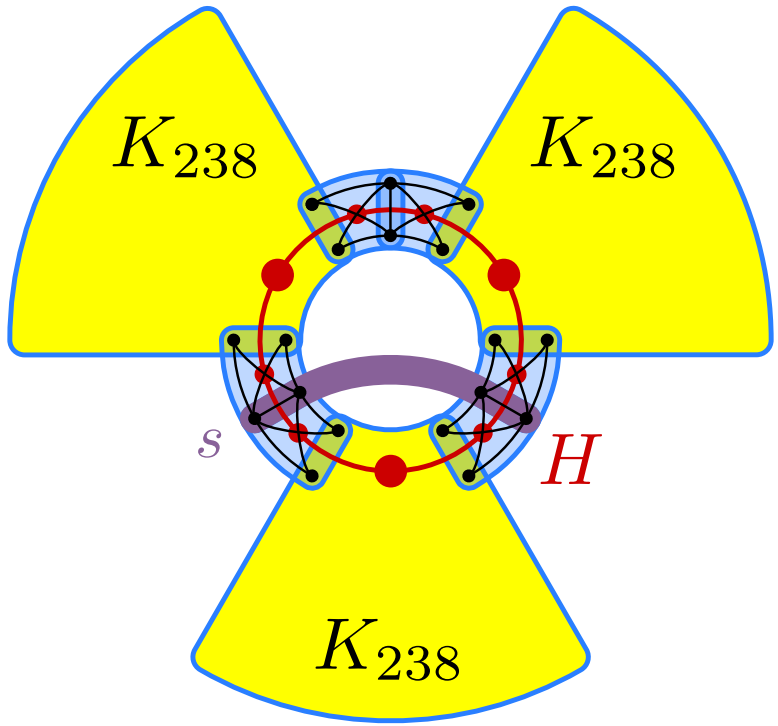
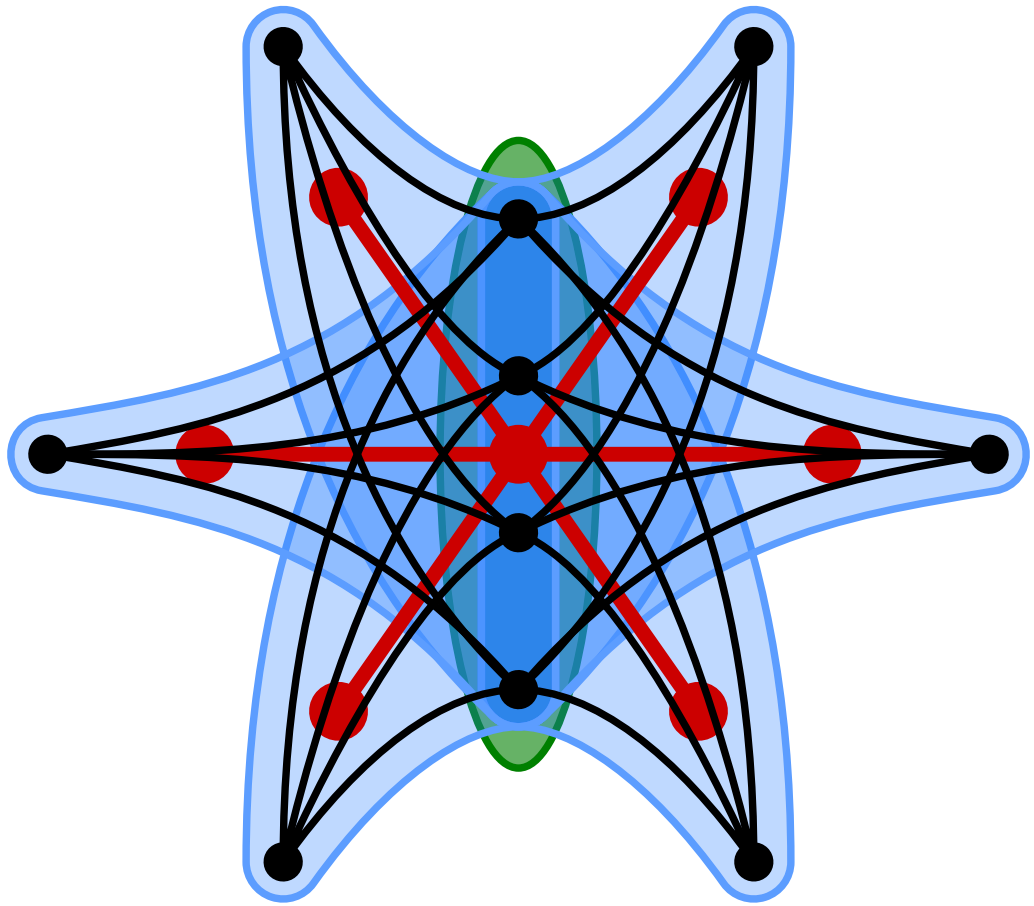
$(H, \mathcal{V})$ is a *k-daisy* if

- every displayed separation has order $\leqslant k$, and
- every induced separation of order $\leqslant k$ is also displayed.



A *k-anemone* is a star-decomposition of G such that

- central bag has size $\leqslant k$, and
- every leaf-bag includes the central bag.



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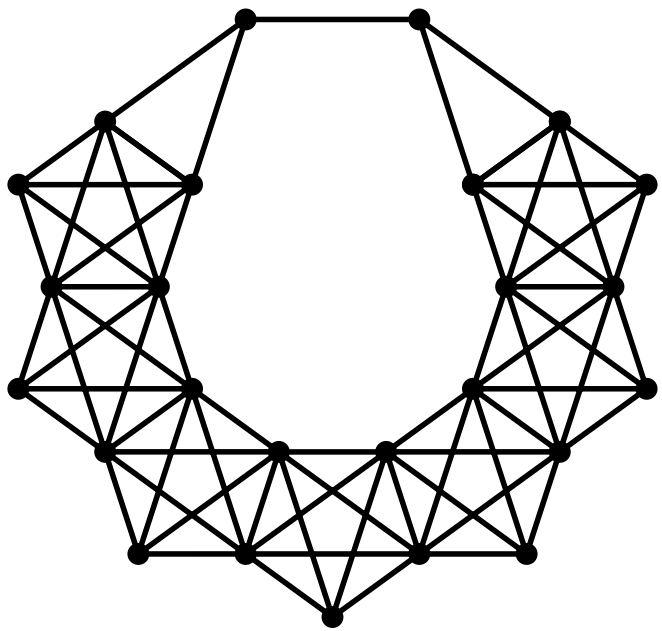

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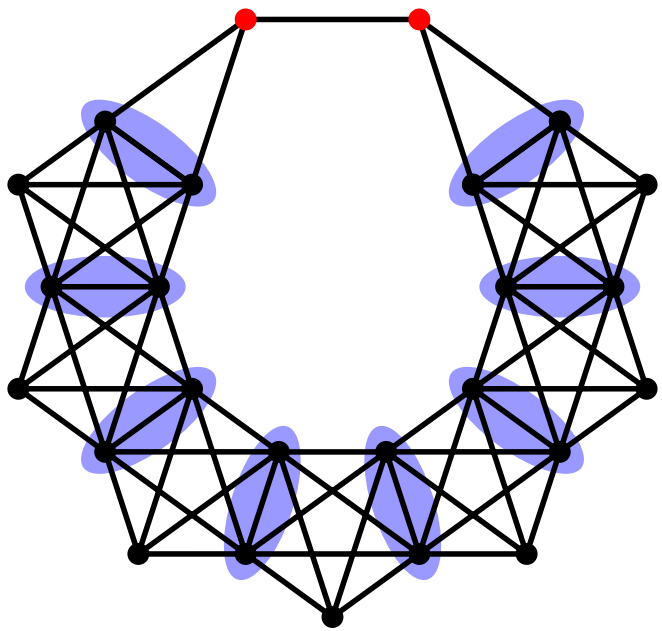

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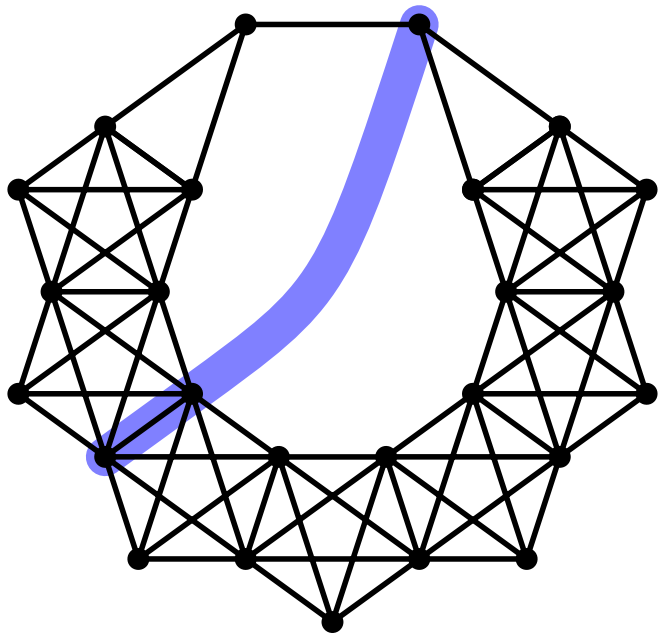

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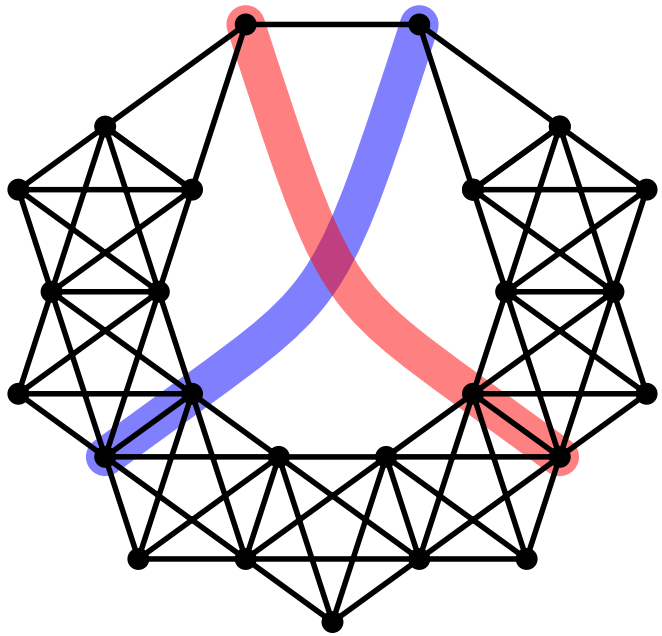
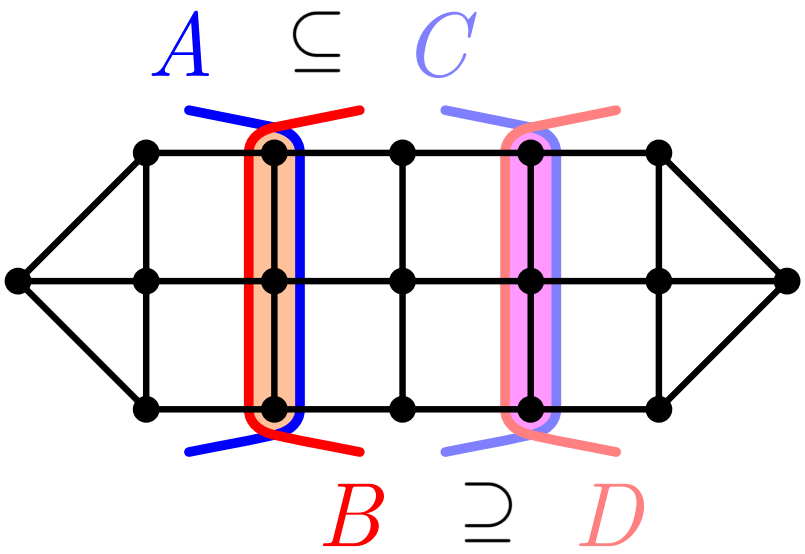
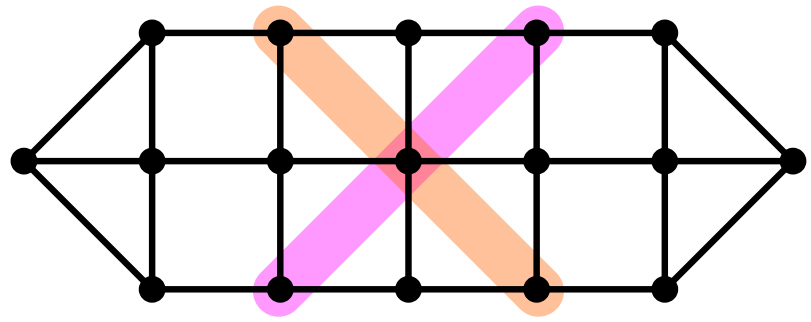


$(A, B) \leqslant (C, D) :\Longleftrightarrow A \subseteq C$ and $B \supseteq D$

(A, B) and (C, D) are *nested* $:\Longleftrightarrow (A, B) \leqslant (C, D)$ after possibly swapping names A with B or C with D

cross $:\Longleftrightarrow$ not nested



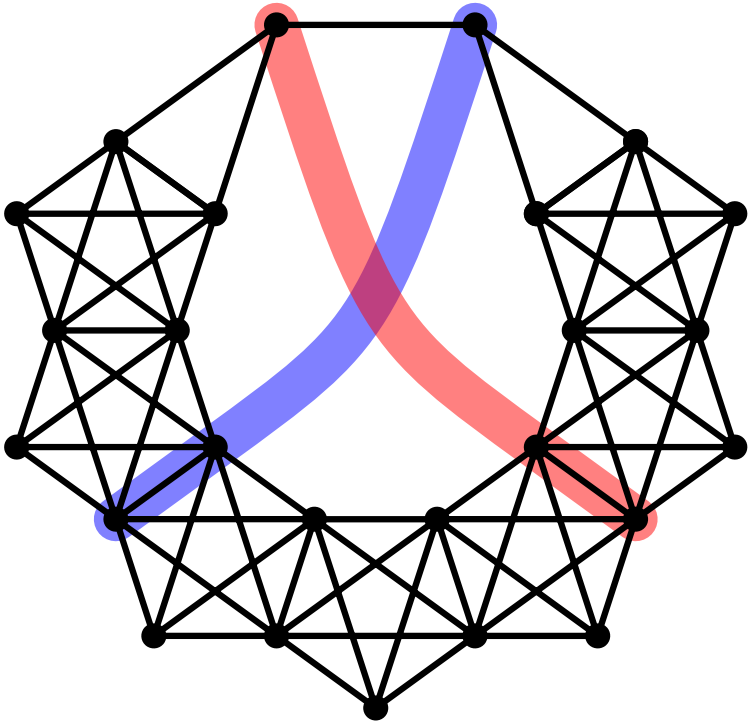


$(A, B) \leqslant_\varepsilon (C, D) :\Longleftrightarrow A \subseteq_\varepsilon C$ and $B \supseteq_\varepsilon D$, where

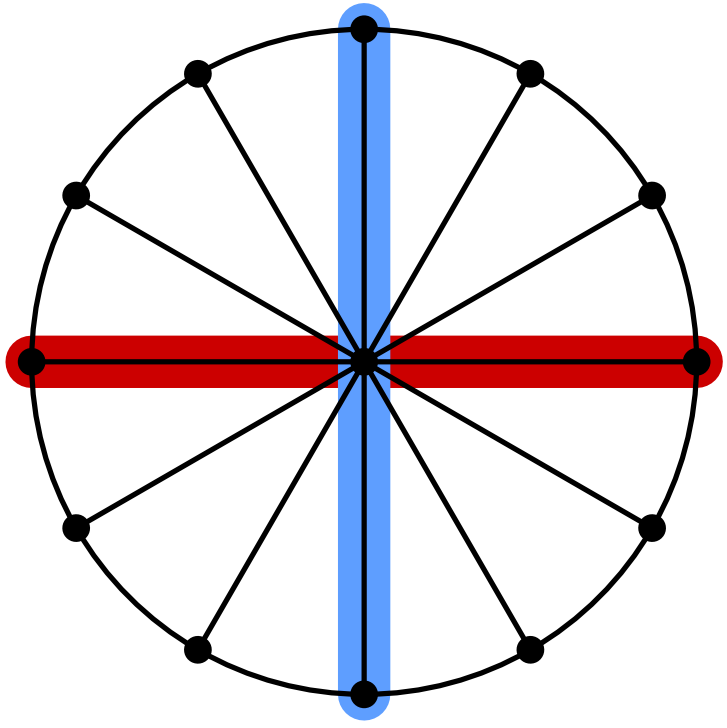
$X \subseteq_\varepsilon Y :\Longleftrightarrow |X \smallsetminus Y| \leqslant \varepsilon$

(A, B) and (C, D) are ε-nested $:\Longleftrightarrow (A, B) \leqslant_\varepsilon (C, D)$ after possibly swapping names A with B or C with D

ε-cross $:\Longleftrightarrow$ not ε-nested



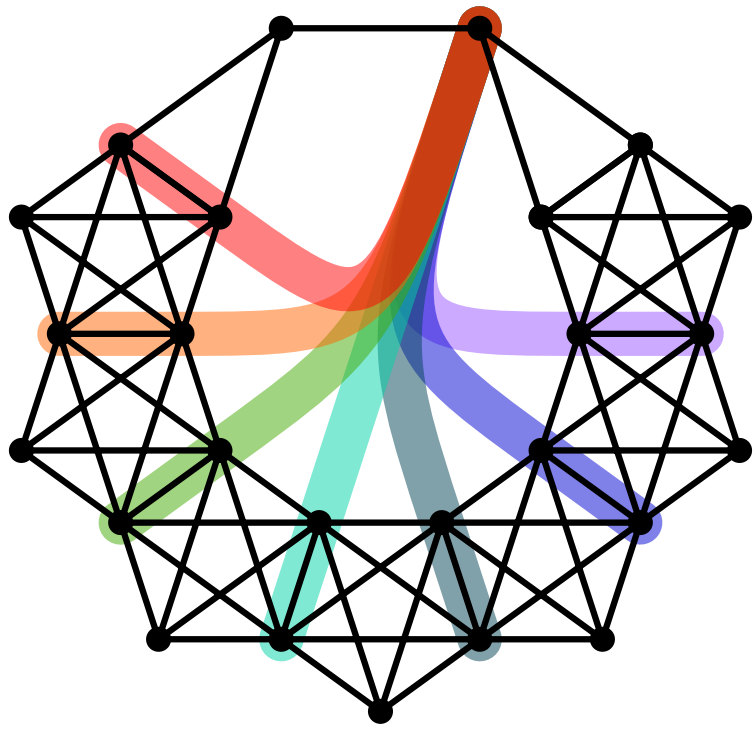
1-nested



2-crossed

A $(\leqslant k)$-separation is *totally-ε-nested* $:\Longleftrightarrow$

it is ε-nested with every $(\leqslant k)$-separation (of G)

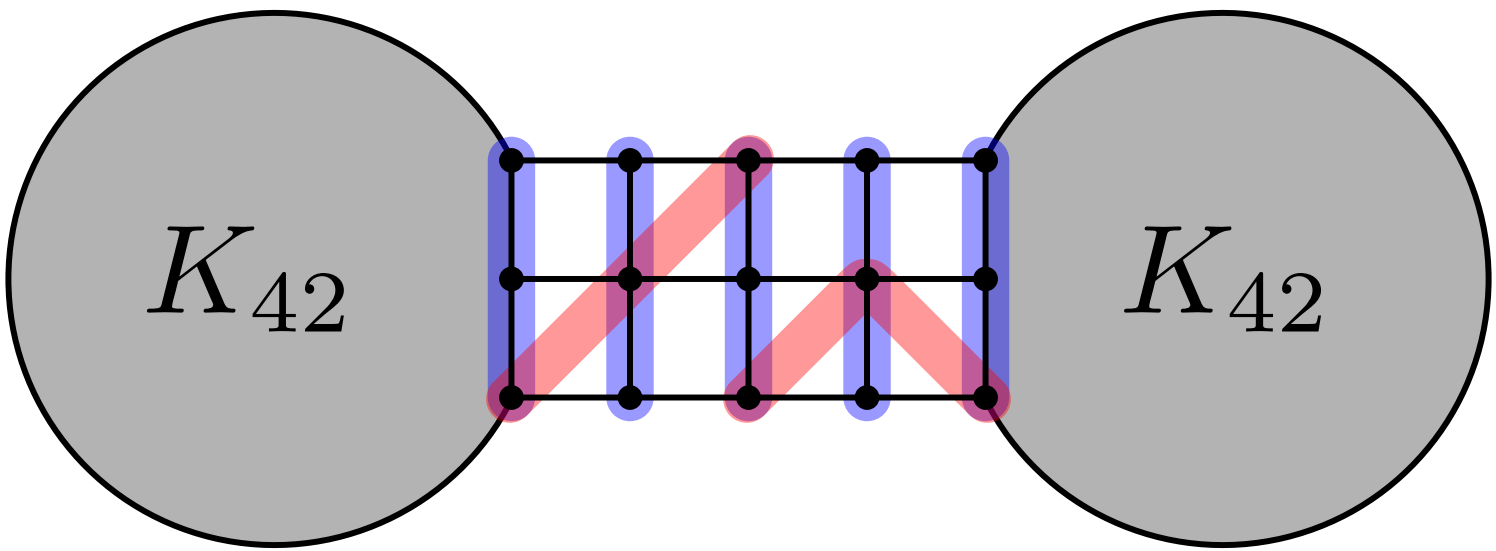


totally-1-nested

Wishful Thinking: $\forall q, k \in \mathbb{N}\ \exists q', \varepsilon \in \mathbb{N}$:

Every q-almost-k-con'd G has a tree-decomposition $(T, \mathcal{V})$ along totally-ε-nested $(\leqslant k)$-separations such that:

- every torso at a $t \in V(T)$ is q'-almost-$(k+1)$-con'd or $(T, \mathcal{V})$ is ε-refined at t by a k-flower F_t;
- every k-separation of G of breadth $\geqslant \varepsilon$ is ε-close to a separation of $(T, \mathcal{V})$ or to a displayed separation of some F_t;
- $(T, \mathcal{V})$ is ε-canonical.



Main Result (Bourneuf, Carmesin, K. and Planken 26+)

$\forall q, k \in \mathbb{N}\ \exists q', \varepsilon \in \mathbb{N}$:

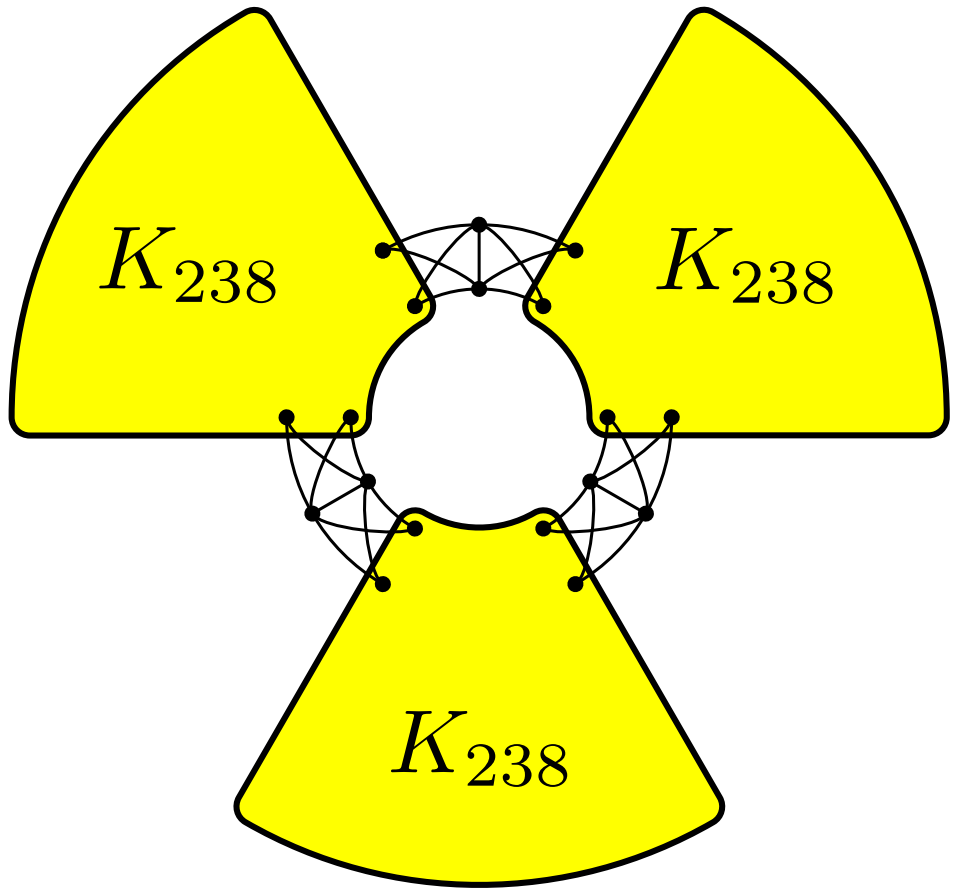
Every q-almost-k-con'd G has a tree-decomposition $(T, \mathcal{V})$
along totally-ε-nested $(\leqslant k)$-separations such that:

- every torso at a $t \in V(T)$ is q'-almost-$(k+1)$-con'd
  or $(T, \mathcal{V})$ is ε-refined at t by a k-flower F_t;
- every k-separation of G of breadth $\geqslant \varepsilon$ is ε-close to
  a separation of $(T, \mathcal{V})$ or to a displayed separation of some F_t;
- $(T, \mathcal{V})$ is ε-canonical.

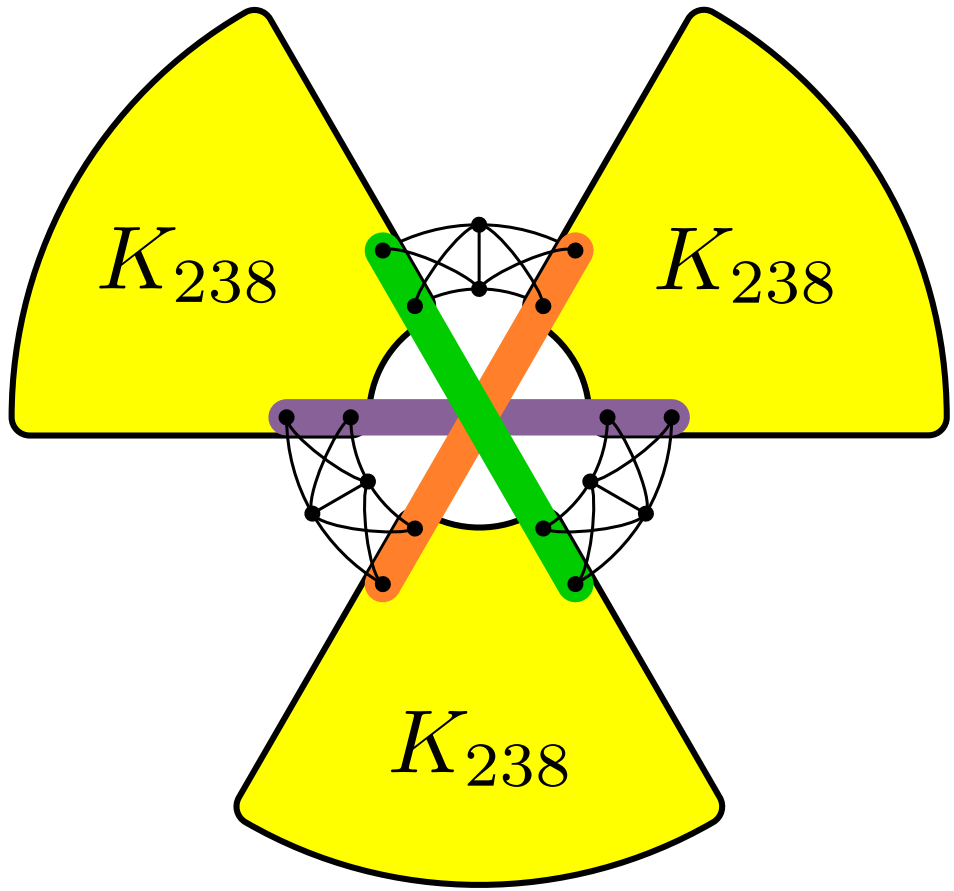
Proof: 100+ pages

Referees:

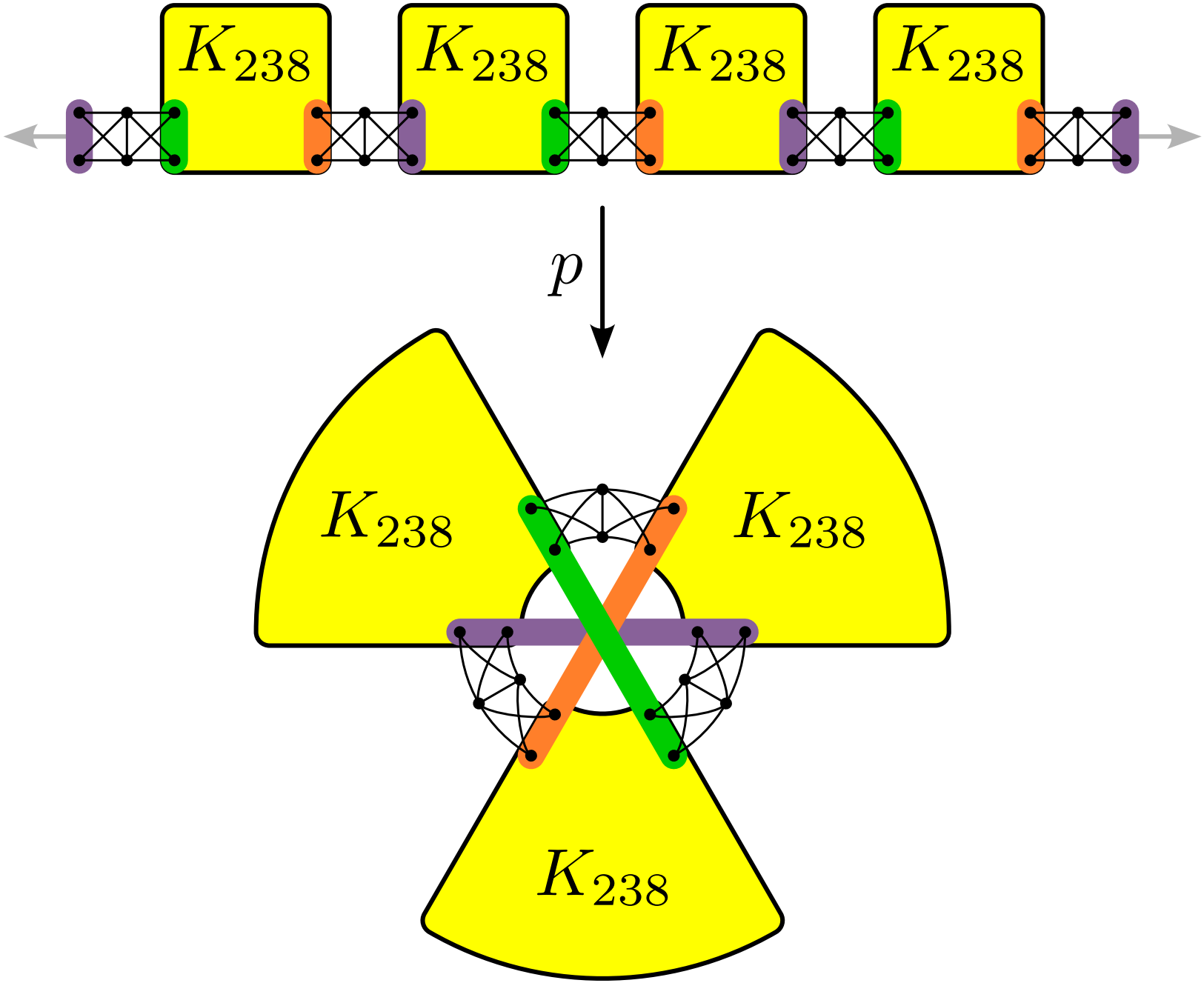
Key idea: Coverings



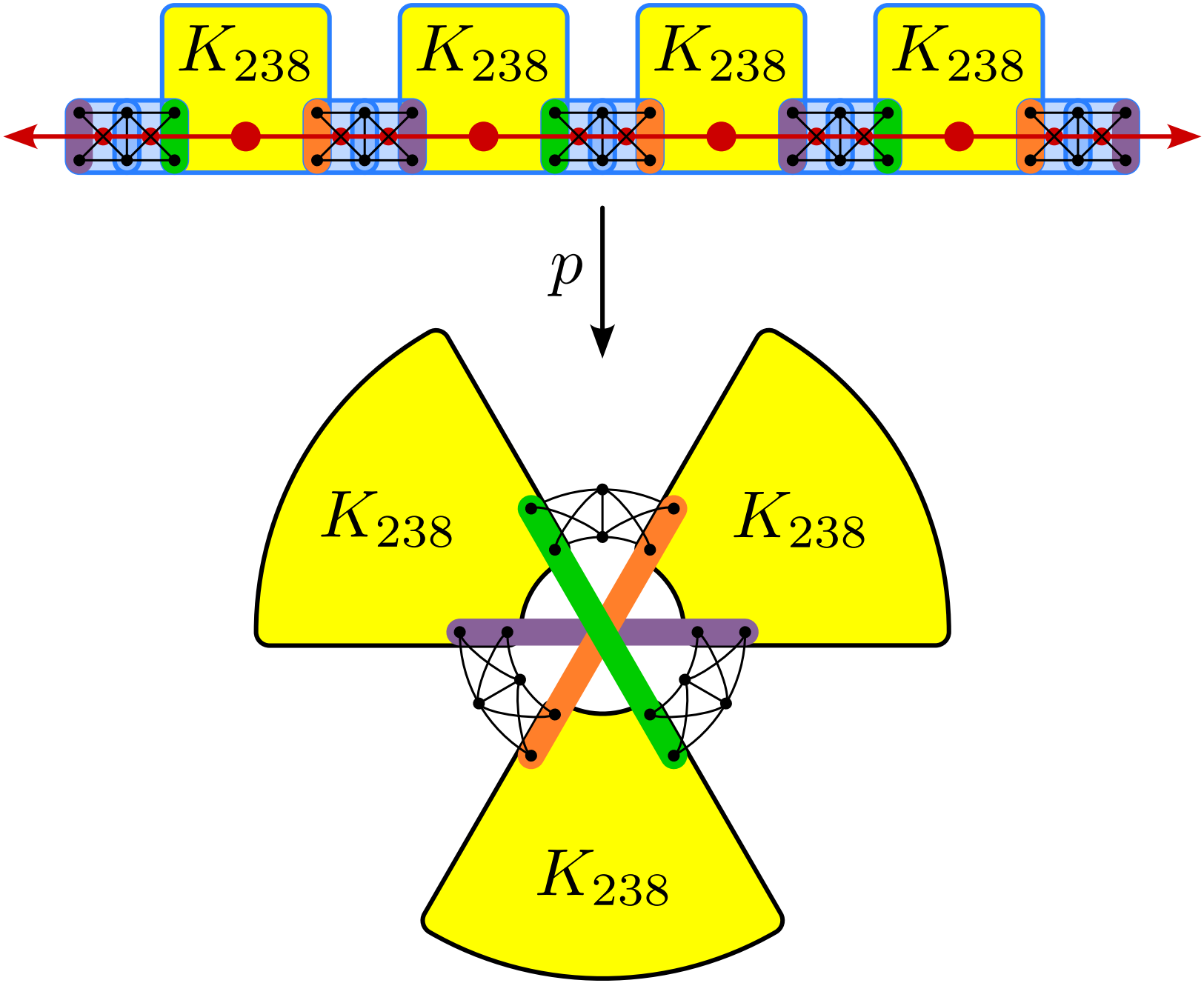
Key idea: Coverings



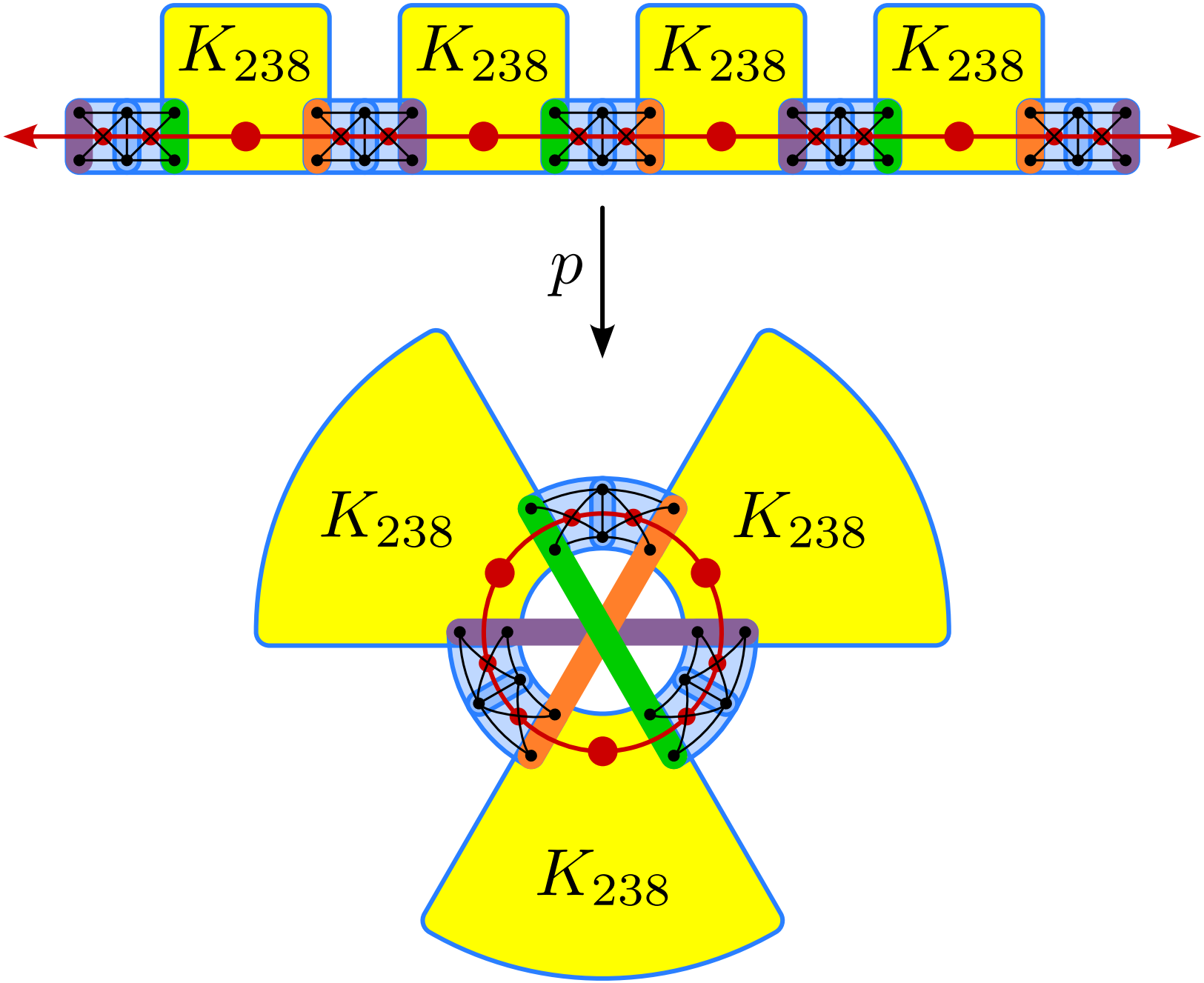
Key idea: Coverings



Key idea: Coverings



Key idea: Coverings



Open: Find **exact** Tutte-decomposition for arbitrary connectivity.

Open: Directed graphs?

$k = 1$: Bowler, Gut, Hatzel, Kawarabayashi, Muzi, Reich 23

$k \geqslant 2$: ???

Open: Matroids for $k \geqslant 4$.

Main Result (Bourneuf, Carmesin, K. and Planken 26+)

$\forall q, k \in \mathbb{N}\ \exists q', \varepsilon \in \mathbb{N}$:

Every q-almost-k-con'd G has a tree-decomposition $(T, \mathcal{V})$ along totally-ε-nested $(\leqslant k)$-separations such that:

- every torso at a $t \in V(T)$ is q'-almost-$(k+1)$-con'd or $(T, \mathcal{V})$ is ε-refined at t by a k-flower F_t;
- every k-separation of G of breadth $\geqslant \varepsilon$ is ε-close to a separation of $(T, \mathcal{V})$ or to a displayed separation of some F_t;
- $(T, \mathcal{V})$ is ε-canonical.

Open: Canonical for graphs. Digraphs for $k \geqslant 2$. Matroids for $k \geqslant 4$.

arXiv: soon

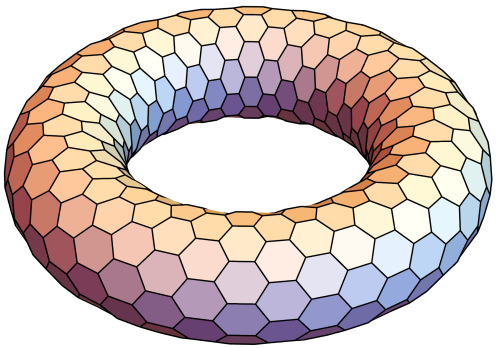
Thank you :)

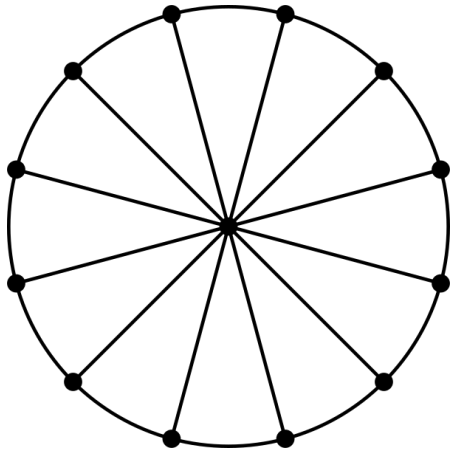
Slides: jan-kurkofka.eu

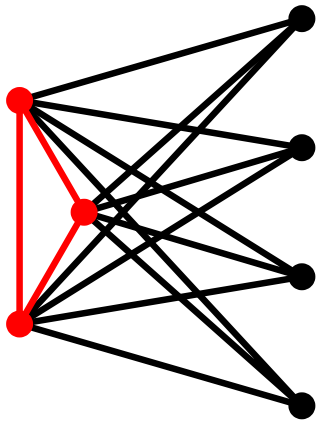
Theorem (Carmesin & K. 23)

Every 3-con'd G decomposes along its totally-nested nontrivial tri-separations into parts that are

- quasi-4-con'd
- wheels
- thickened $K_{3,m}$







Theorem (K. & Planken 25)

Every 4-con'd G decomposes along its totally-nested tetra-separations into parts that are

- quasi-5-con'd
- generalised double-wheels
- thickened $K_{4,m}$
- cycles of triangles and 3-con'd graphs on $\leqslant 5$ vxs.