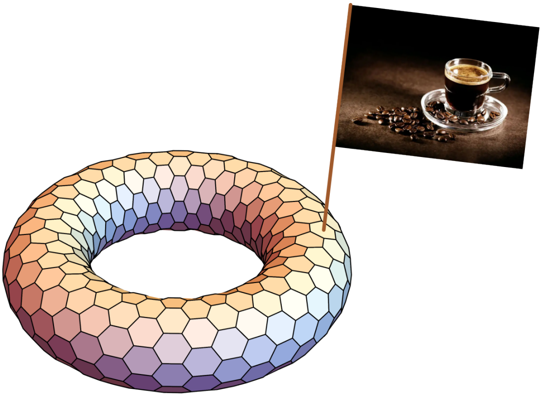
A Tutte-type canonical decomposition of 3- and 4-connected graphs

Jan Kurkofka (TU Freiberg)



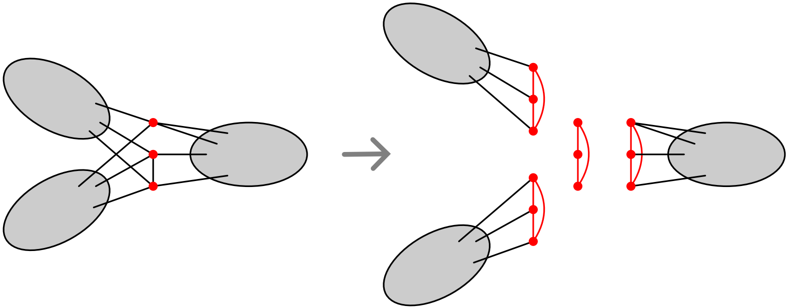
Joint work with Tim Planken

BWAG '25

Problem: Canonically decompose k-con'd G along k-separators into parts that are $(k+1)$-con'd or 'basic'.

canonical $:\Longleftrightarrow$ $\mathrm{Aut}(G)$-invariant

Decomposing G along a k-separator:



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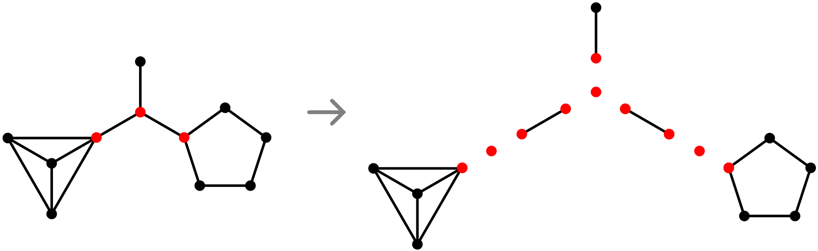
$k = 1$:



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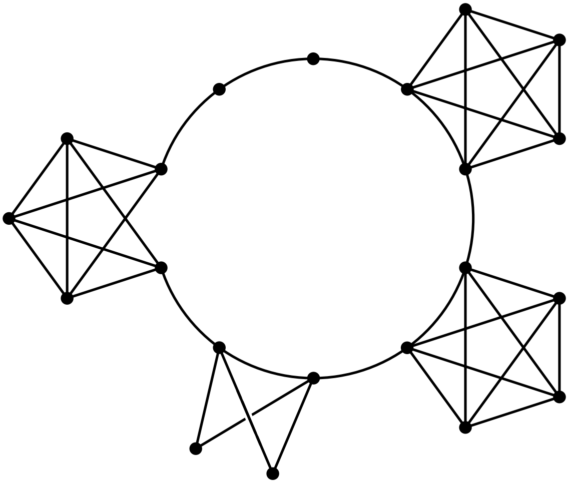

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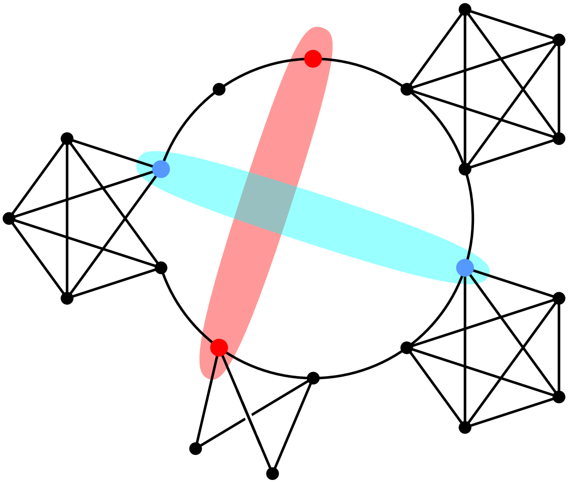

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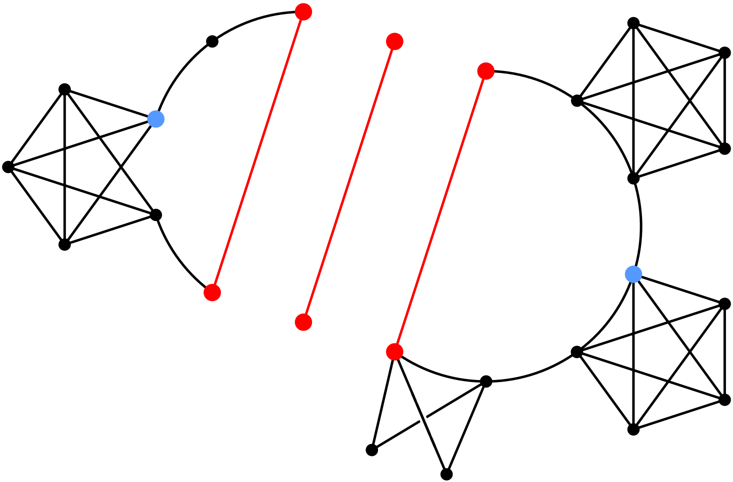
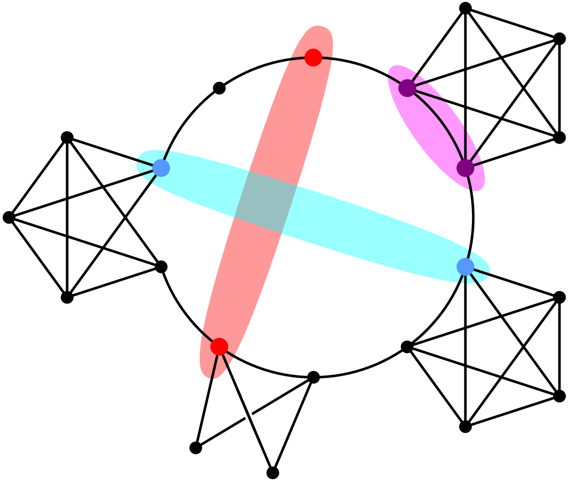


Two k-separators *cross* if they separate each other;

otherwise they are *nested*.



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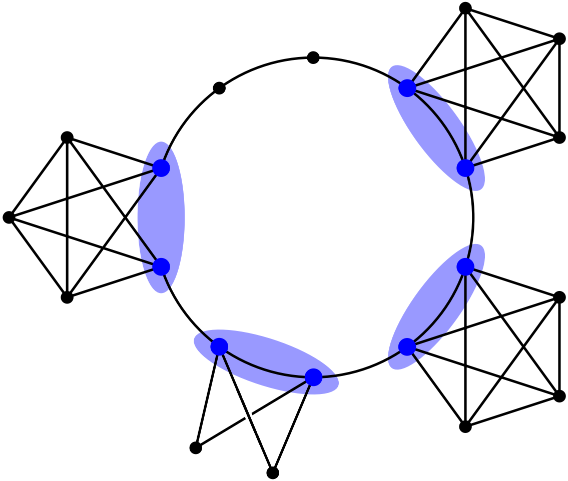
A k-separator is *totally-nested* if it is nested with every k-separator.



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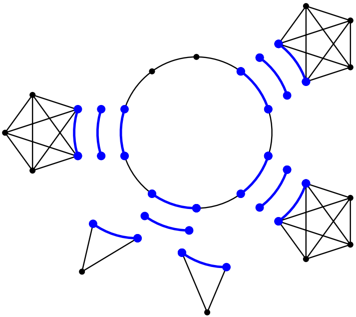

Theorem (Tutte 66)

Every 2-con'd G decomposes along its totally-nested 2-separators into 3-con'd graphs, cycles and K_2's.

Guess: Every k-con'd G decomposes along its totally-nested k-separators into $(k+1)$-con'd graphs and 'basic' graphs.

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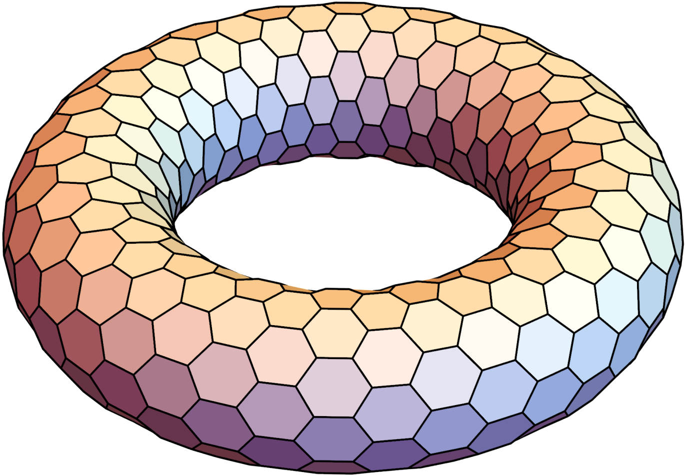
Challenge 1 (Figure: $k = 3$)



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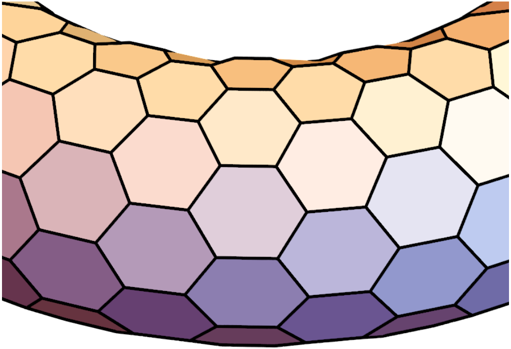

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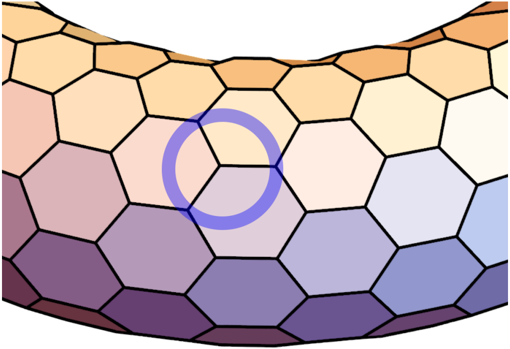

(set of all k-separators) $= \{ N(v) : v \in V(G) \}$

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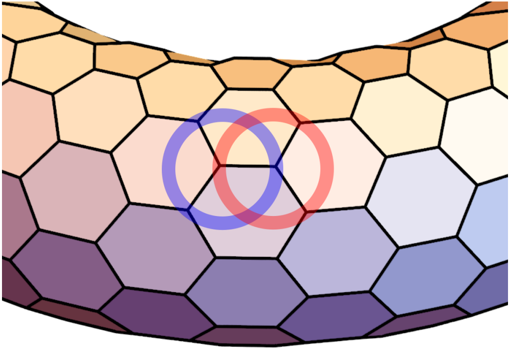

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$\Longrightarrow$ for every edge uv: $N(u)$ crosses $N(v)$.

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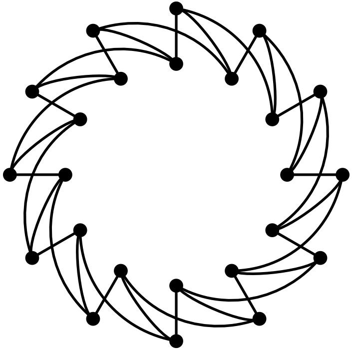

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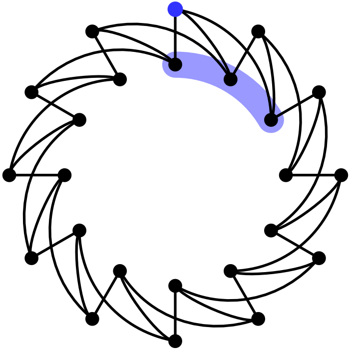

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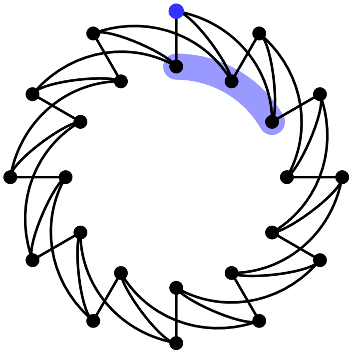

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Guess: Every k-con'd G decomposes along its totally-nested k-separators into **quasi**-$(k+1)$-con'd graphs and 'basic' graphs.

$\Longleftrightarrow$ k-con'd and every k-sep'r cuts off only one vertex

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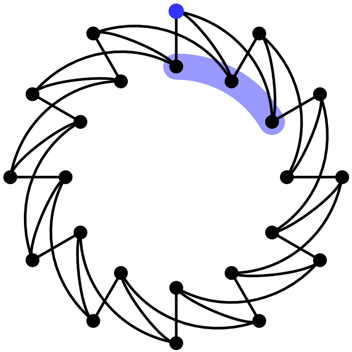

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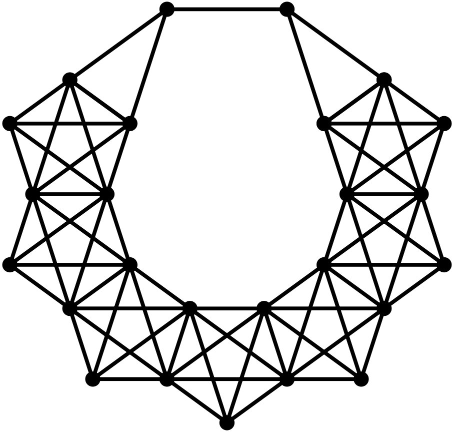

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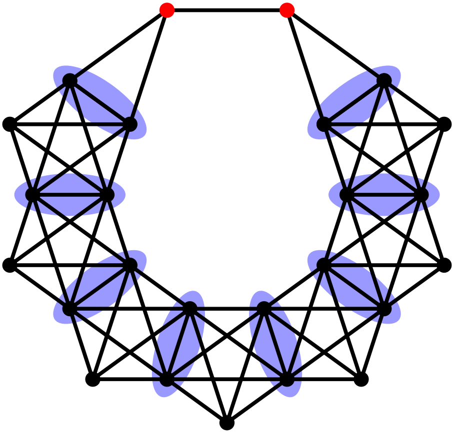

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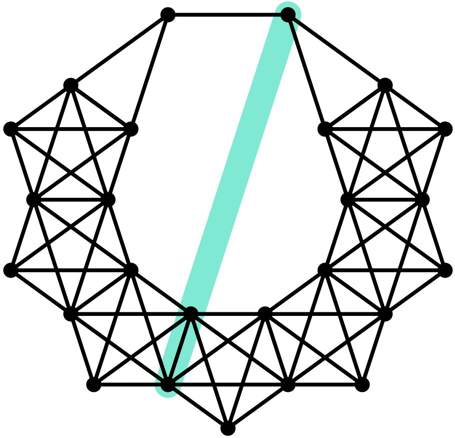

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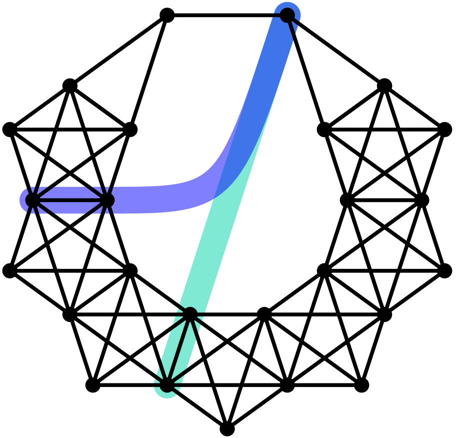

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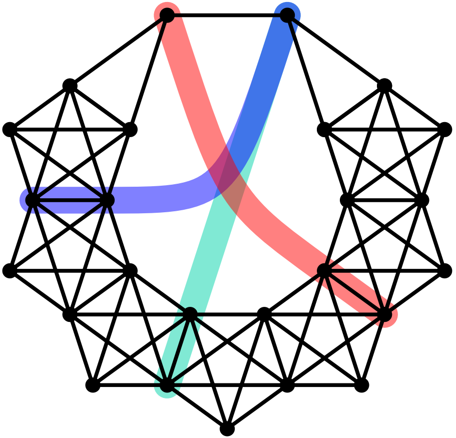

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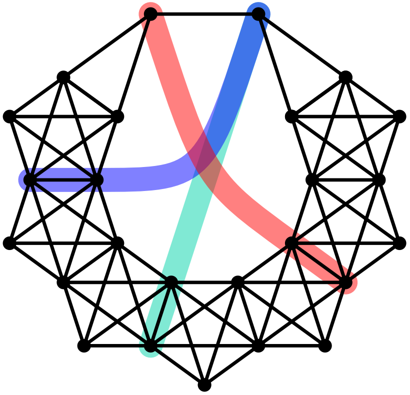

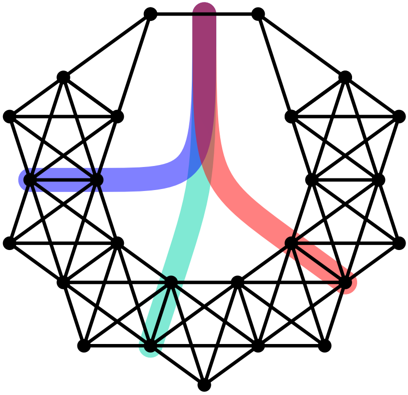

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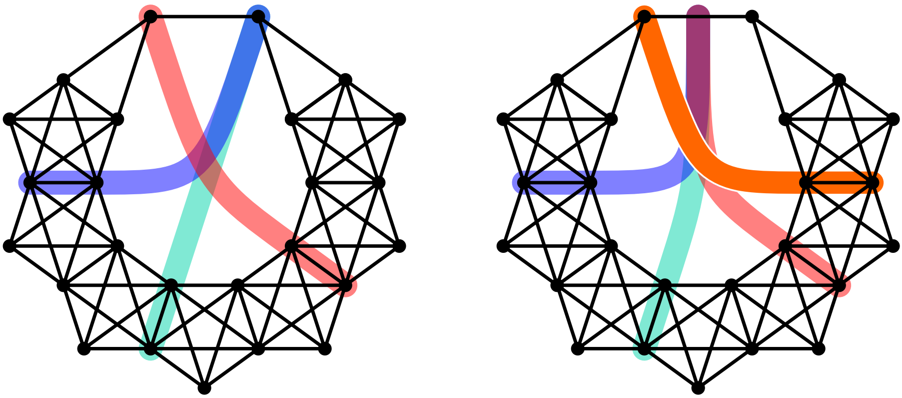

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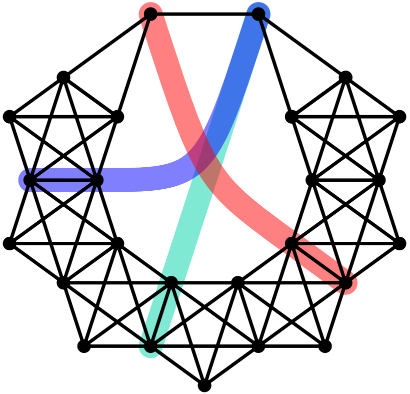

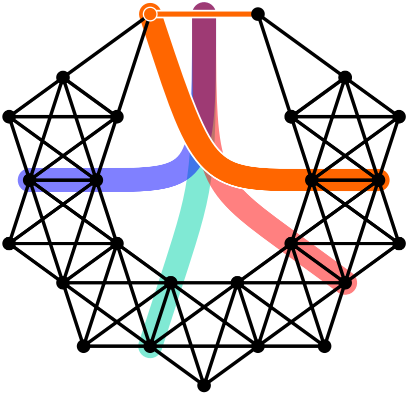

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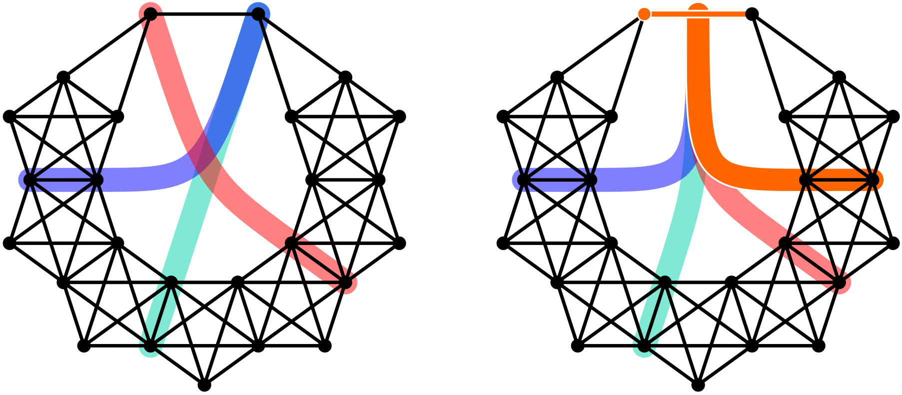

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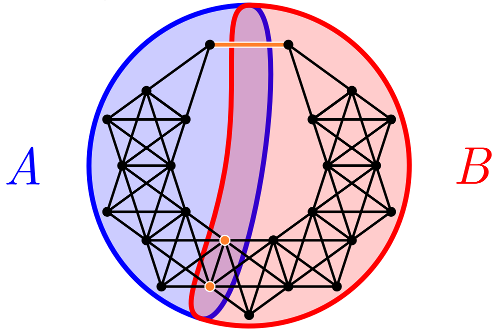

mixed-separation of G: (A, B) with $A \cup B = V(G)$ and $A, B \neq V(G)$

separator of (A, B): $(A \cap B) \cup E(A \setminus B, B \setminus A)$

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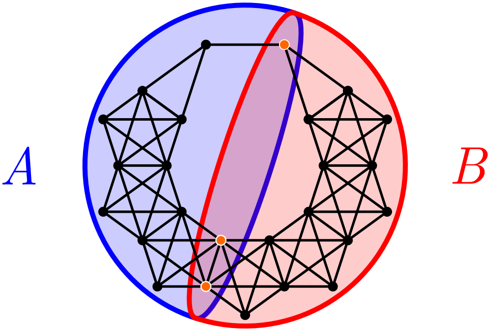

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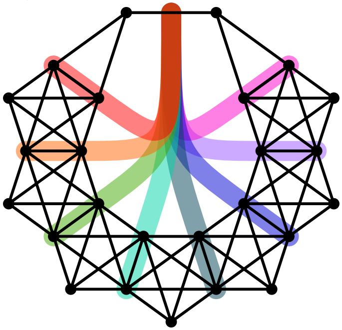

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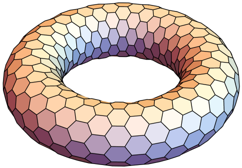
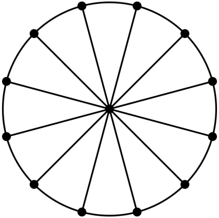
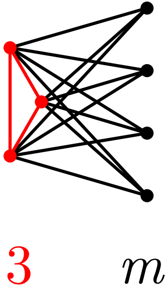
Theorem (Carmesin & K. 23)

Every 3-con'd G decomposes along its totally-nested nontrivial tri-separations into parts that are

- quasi-4-con'd
- wheels
- thickened $K_{3,m}$







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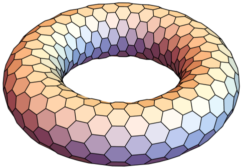
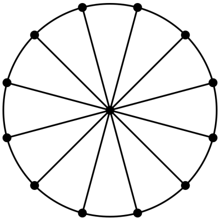
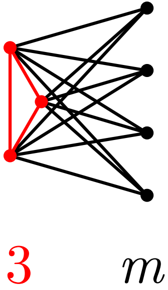


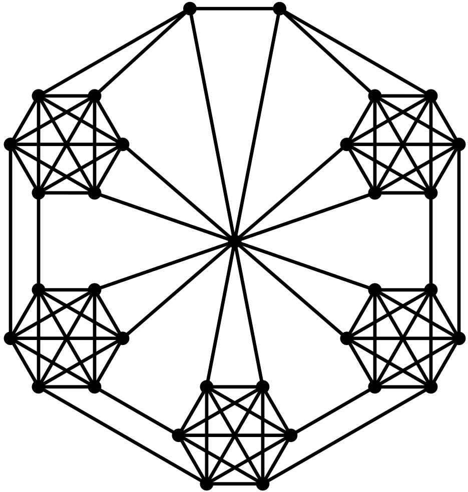




4-connectivity?

Guess: Every k-con'd G decomposes along its totally-nested k-separators into quasi-$(k+1)$-con'd graphs and 'basic' graphs.

Challenge 2: Verbatim extension of tri-separations to $k = 4$



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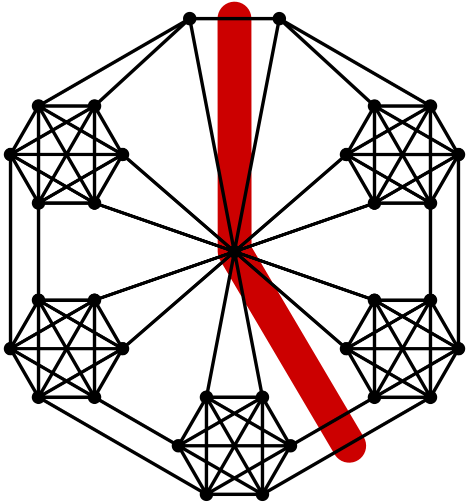

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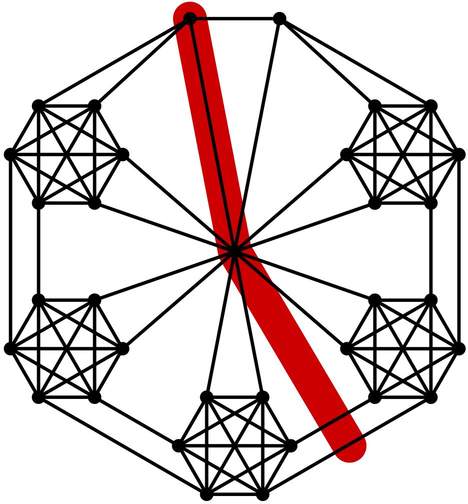

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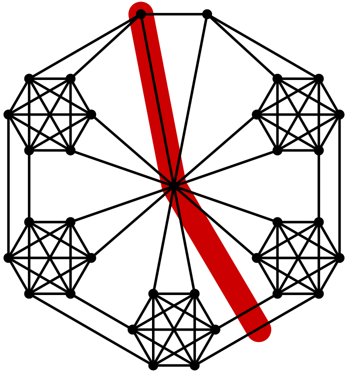
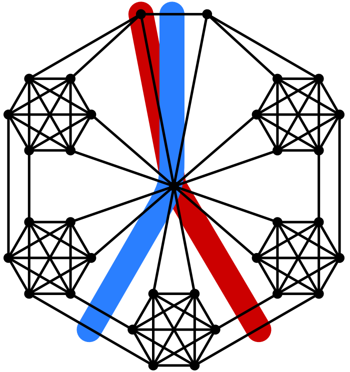


A tri-separation of G is a mixed-sep'n (A, B) with $|\text{sep'r}| = 3$ s.t. every vx in $A \cap B$ has $\geqslant 2$ neighb's in A and B.

Guess: Every k-con'd G decomposes along its totally-nested k-separators into quasi-$(k+1)$-con'd graphs and 'basic' graphs.

Challenge 2: Verbatim extension of tri-separations to $k = 4$ **fails!**



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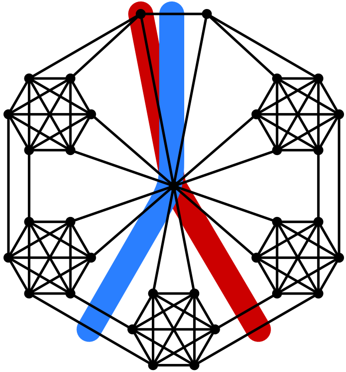

A **tetra-separation** of G is a mixed-sep'n (A, B) with $|\text{sep'r}| = 4$ s.t.:

- every vx in $A \cap B$ has $\geqslant 2$ neighb's in $A \setminus B$ and $B \setminus A$
- the edges in the sep'r form a matching

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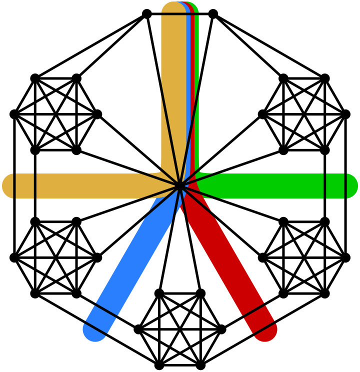

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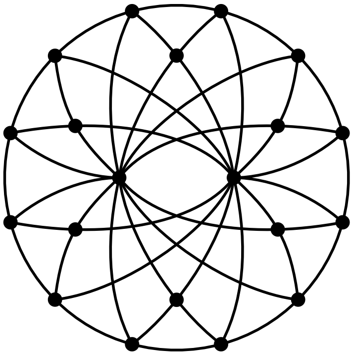
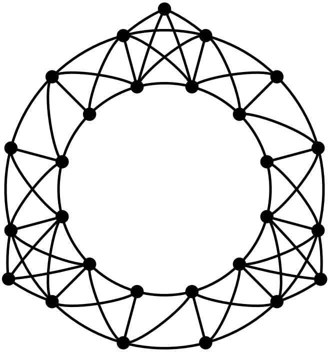
Main result (K. & Planken 25)

Every 4-con'd G decomposes along its totally-nested tetra-separations into parts that are

- quasi-5-con'd
- generalised double-wheels
- thickened $K_{4,m}$
- cycles of triangles and 3-con'd graphs on $\leqslant 5$ vxs.

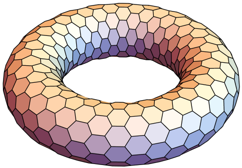


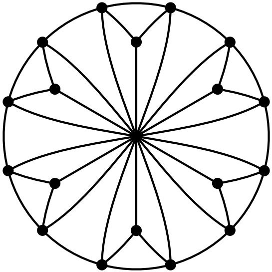


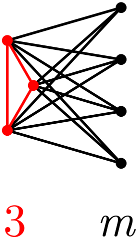
Corollary (K. & Planken 25)

Every 3-con'd G decomposes along its totally-nested **strict tri-separations** into parts that are

- quasi-4-con'd
- **generalised wheels**
- thickened $K_{3,m}$







	con'd	
block-cut	$\Downarrow$	
	2-con'd	K_2, K_1
Tutte	$\Downarrow$	
	3-con'd	cycle, K_2
tri-sep'ns	$\Downarrow$	
	quasi-4-con'd	wheel, thickened K_3

	con'd	
block-cut	$\Downarrow$	
	2-con'd	K_2, K_1
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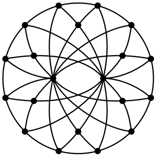
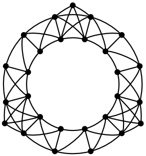
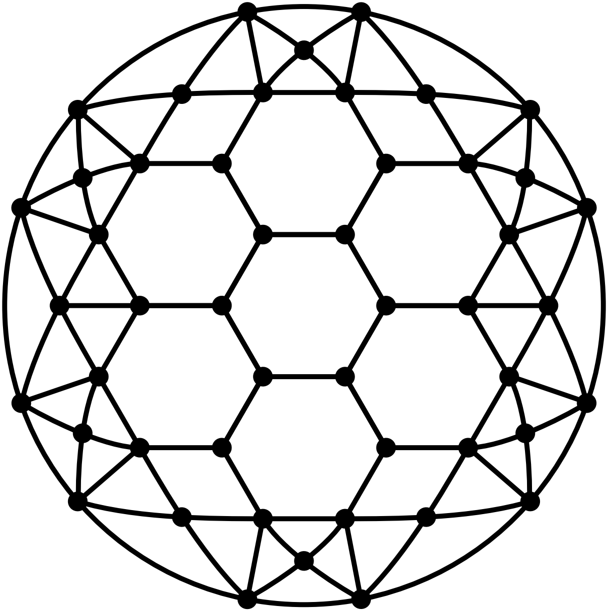
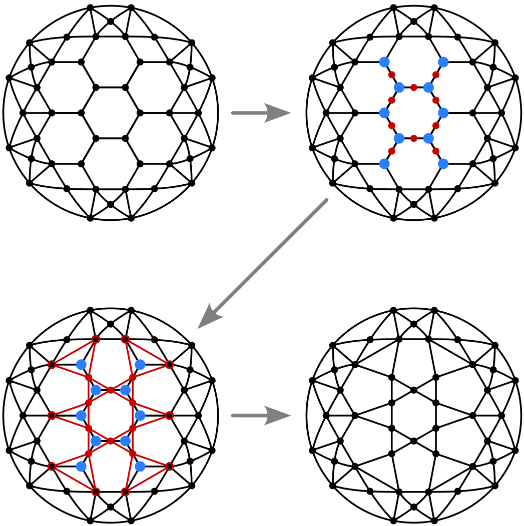
	3-con'd	cycle, K_2
tri-sep'ns	$\Downarrow$	
	quasi-4-con'd	wheel, thickened K_3
tetra-sep'ns	$\Downarrow$	
	quasi-5-con'd	  ...

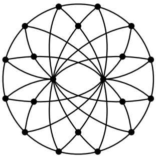
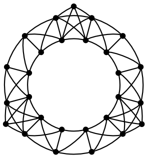


canonical Y–Δ transformation

quasi-4-con'd



4-con'd

	con'd	
block-cut	$\Downarrow$	
	2-con'd	K_2, K_1
Tutte	$\Downarrow$	
	3-con'd	cycle, K_2
tri-sep'ns	$\Downarrow$	
	quasi-4-con'd	wheel, thickened K_3
Y–Δ	$\Downarrow$	
	4-con'd	
tetra-sep'ns	$\Downarrow$	
	quasi-5-con'd	  . . .

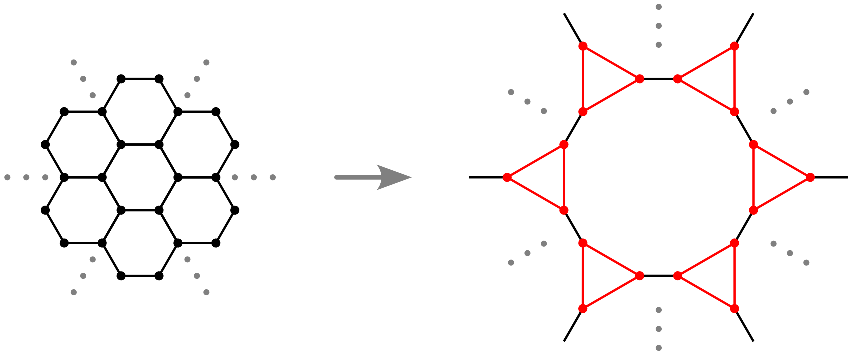
Problem: Classify all **vertex-transitive** finite con'd G

Theorem (Carmesin & K. 23)

Every vertex-transitive finite con'd G is either

- a **cycle**, K_2, K_1,
- quasi-4-con'd or

  K_3-expansion of a 3-regular quasi-4-con'd arc-transitive graph.



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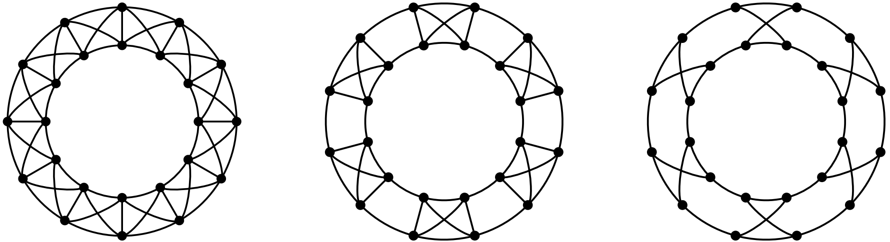
Theorem (K. & Planken 25)

Every quasi-4-con'd vertex-transitive finite G is either

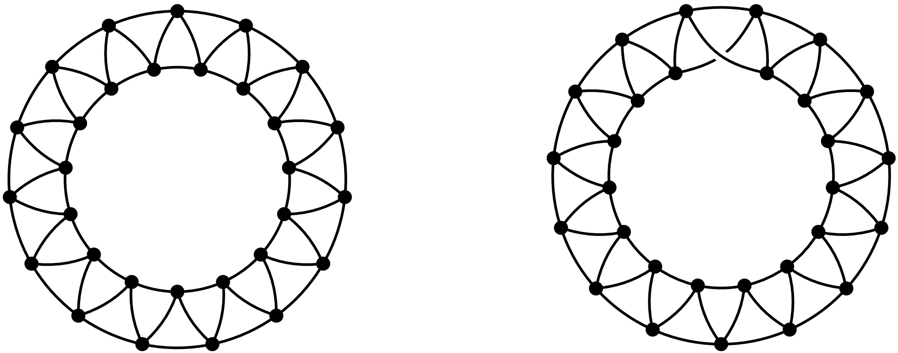
- **bagel-like or cube-like**,
- quasi-5-con'd or

  K_4/C_4-expansion of a 4-regular quasi-5-con'd arc-transitive graph.

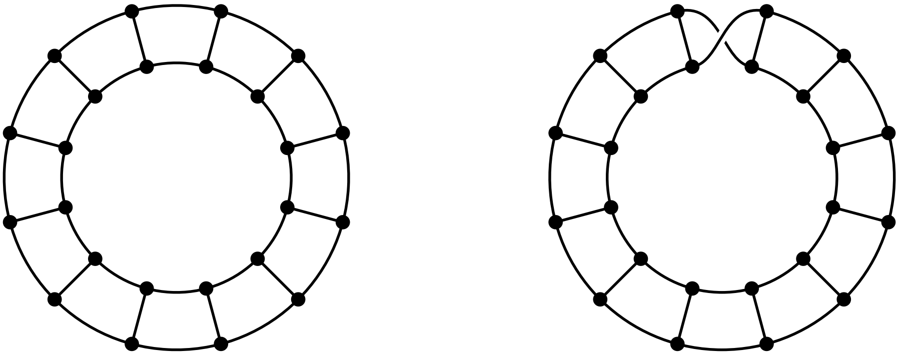
bagel-like



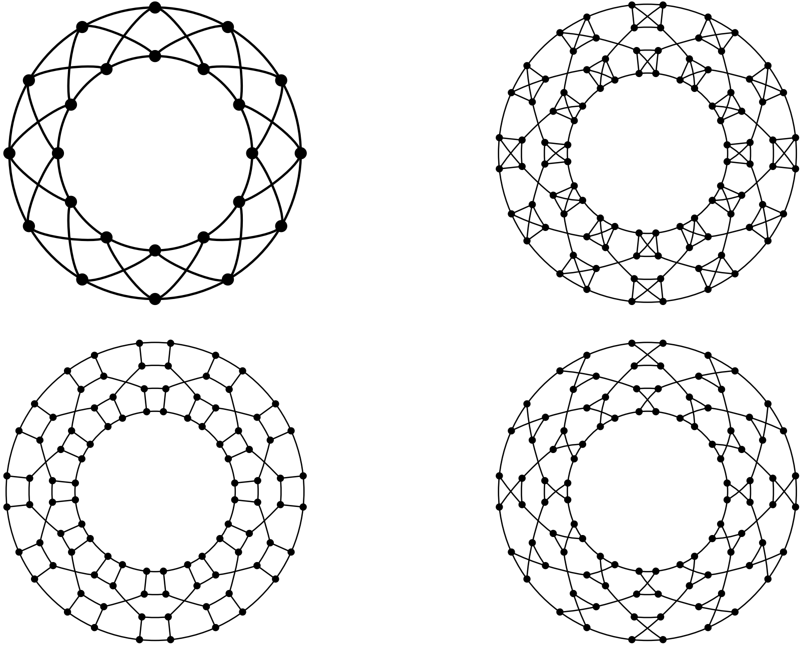
bagel-like



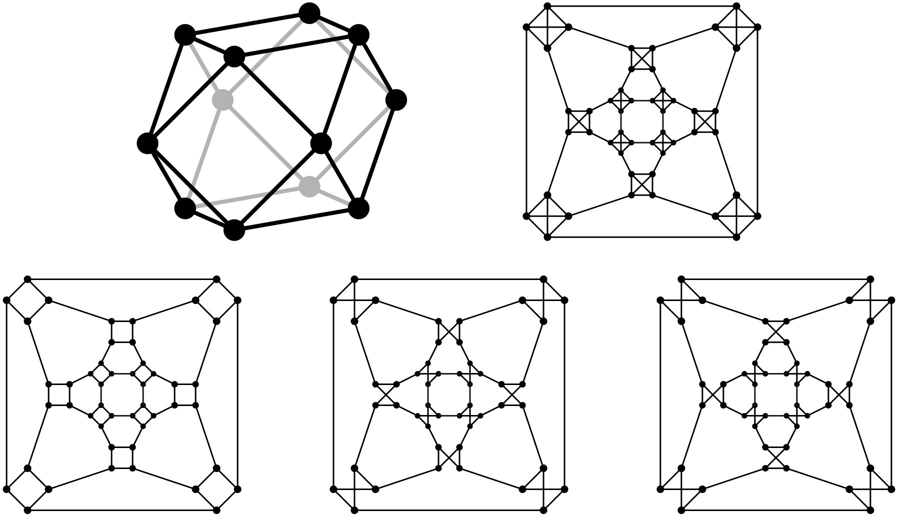
bagel-like



bagel-like



cube-like



Open: Extend Tutte's decomposition to *all* k.

Open: Directed graphs?

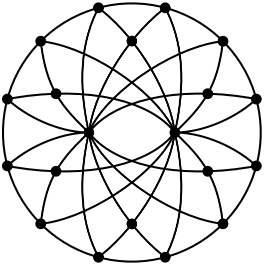
$k = 1$: Bowler, Gut, Hatzel, Kawarabayashi, Muzi, Reich 23

$k \geqslant 2$: ???

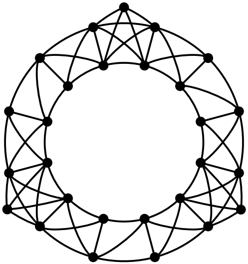
Tetra-separation: mixed-sep'n (A, B) with |sep'r| $= 4$ such that every vx in $A \cap B$ has $\geqslant 2$ neighb's in $A \smallsetminus B$ and in $B \smallsetminus A$, and cross-edges form matching.

Main result (K. & Planken 25)

Every 4-con'd G decomposes along its totally-nested tetra-separations into parts that are quasi-5-con'd, thickened $K_{4,m}$'s,



or



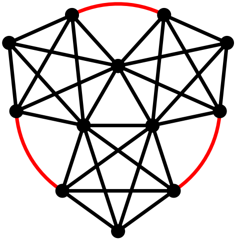
Open: Graphs for $k \geqslant 5$. Digraphs for $k \geqslant 2$.

arXiv: 2504.00760

Thank you :)

Slides: jan-kurkofka.eu

Application: Connectivity Augmentation from 0 to 4



Theorem (Carmesin & Sridharan 25+)

$\exists$ FPT-algorithm with runtime $C(\ell) \cdot \mathrm{Poly}(\,|V(G)|\,)$ and

Input: Graph G, $\ell \in \mathbb{N}$ and $F \subseteq E(\,\overline{G}\,)$

Output: No, or $\leqslant \ell$-sized $X \subseteq F$ such that $G + X$ is 4-con'd

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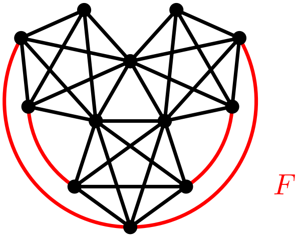

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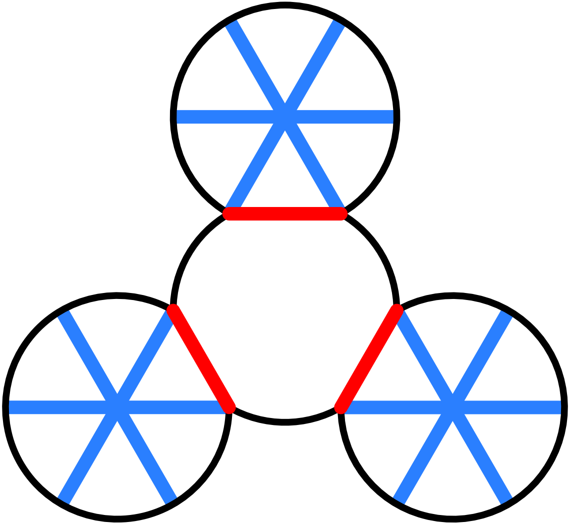
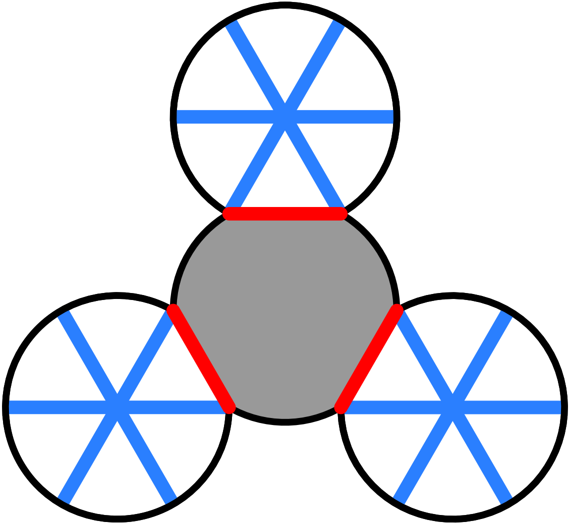
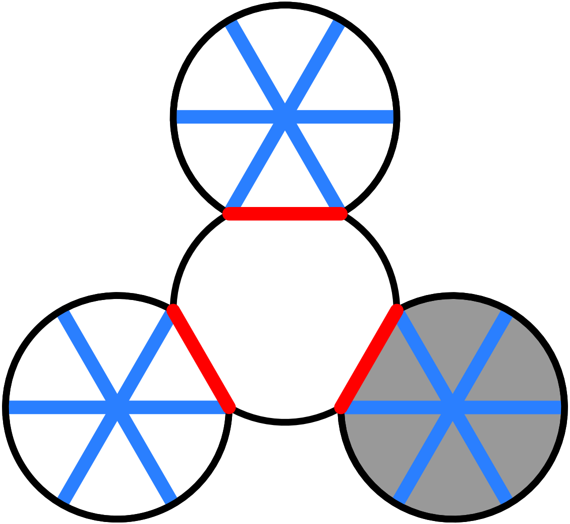
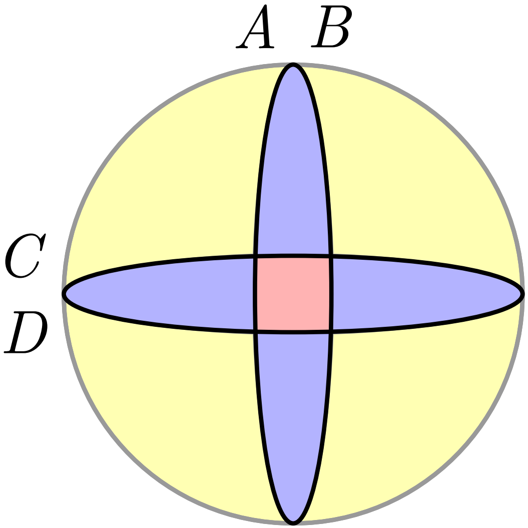
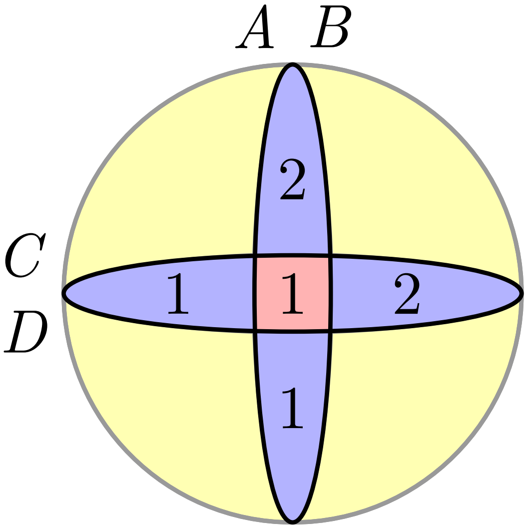
Proof of tetra-decomposition



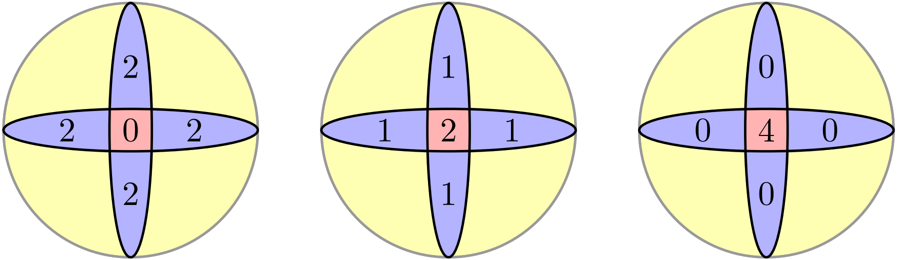




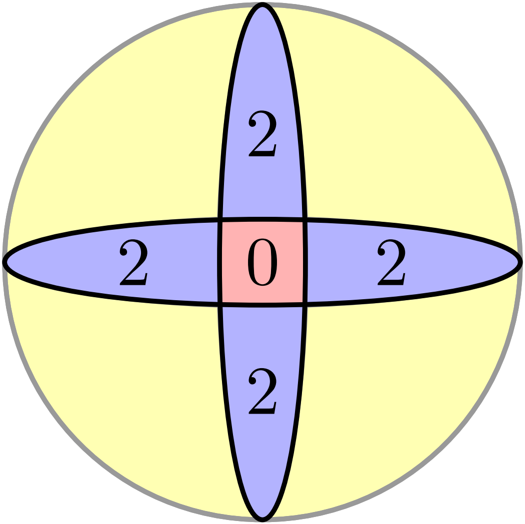

A
B
C
D

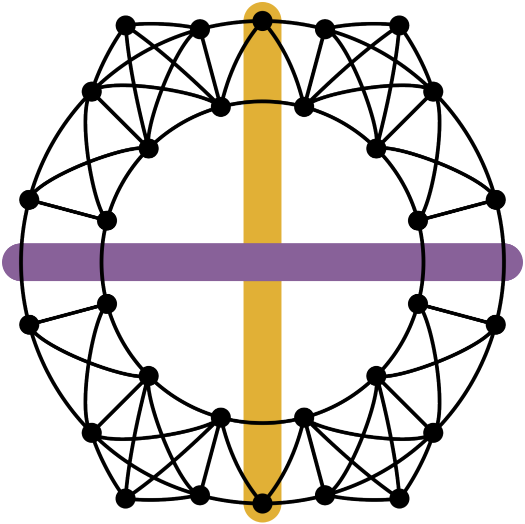

A
B
C
D
2
1
1
2
1

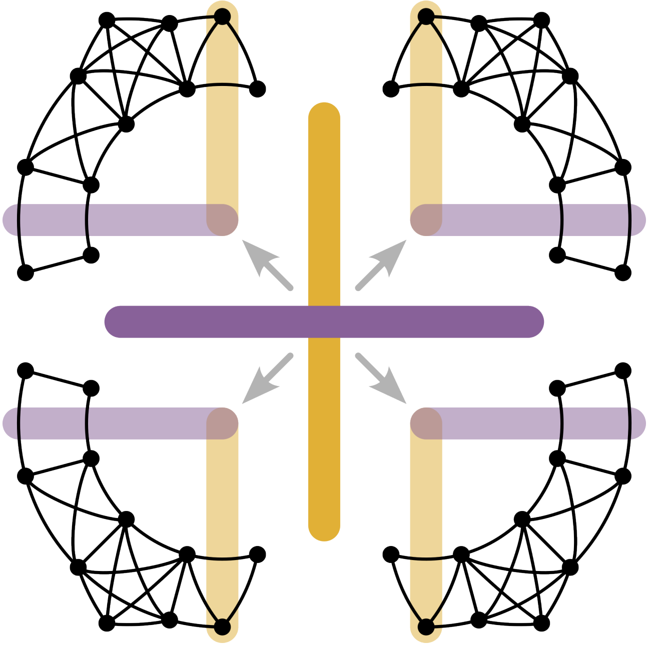
Crossing Lemma. Tetra-sep'ns only cross symmetrically:

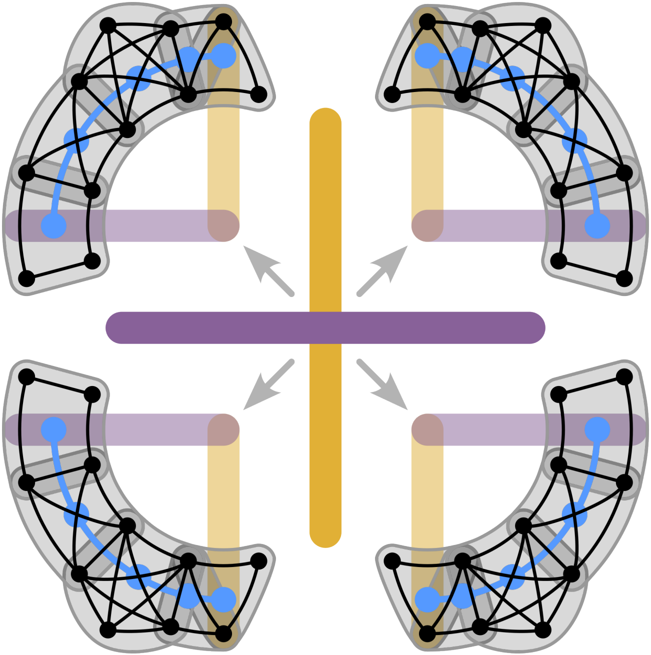


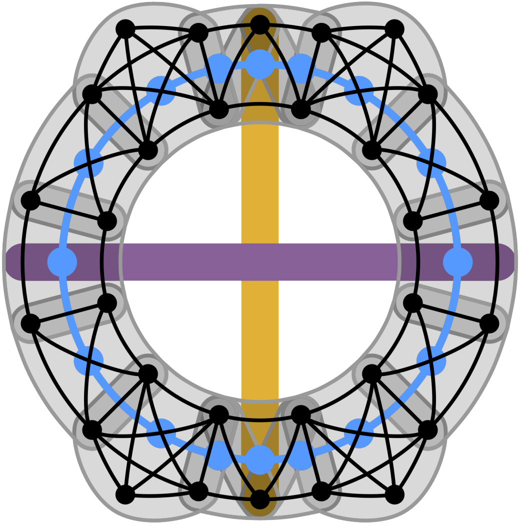
focus

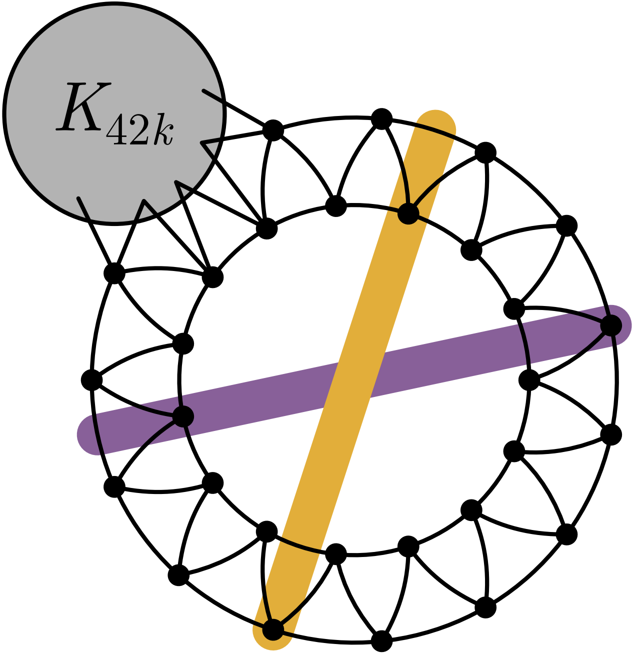


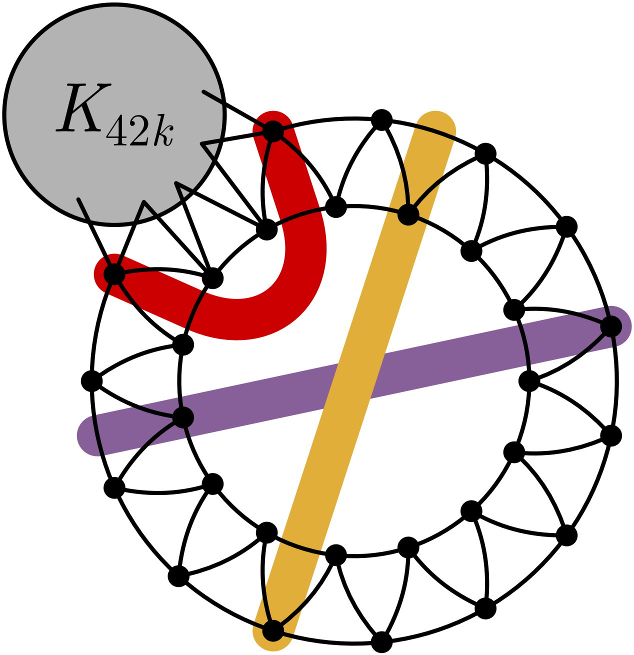






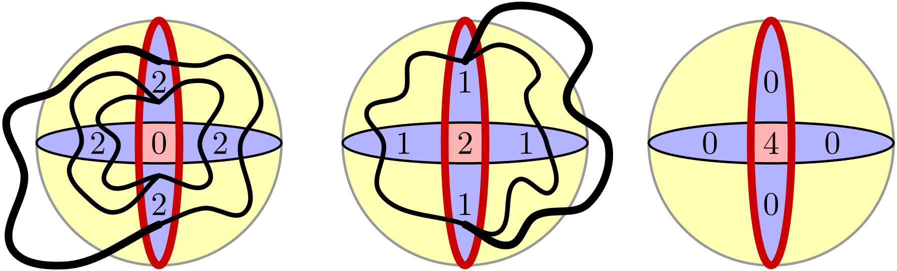


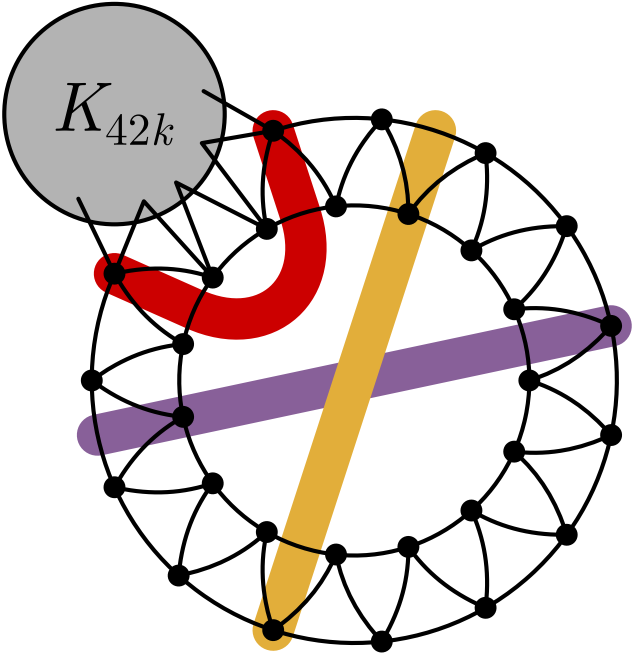

K_{42k}


K_{42k}

(A, B) totally-nested

$\iff$ the sep'r of (A, B) is highly con'd:

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