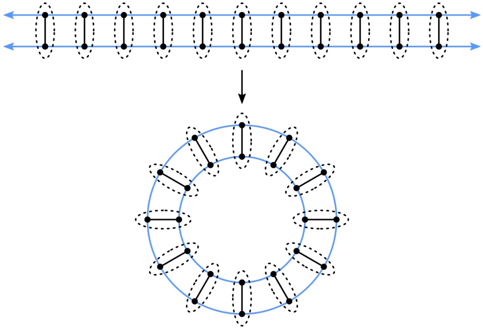
Towards a Stallings-type theorem for finite groups

Jan Kurkofka

TU Freiberg



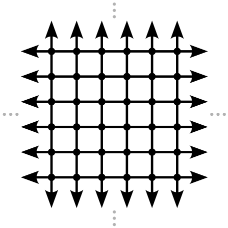
Joint work with

Johannes Carmesin, George Kontogeorgiou and Will J. Turner

Problem. Detect product-structure in groups through combinatorial properties of their Cayley graphs.

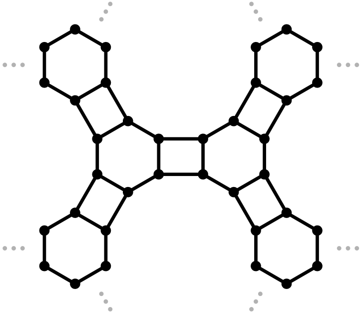
Example. Stallings' Theorem uses *ends* of Caley graphs. . .

end of a graph: equivalence class of 1-way infinite paths w.r.t. the relation ‘not separable by finitely many vertices’



1 end

2 ends



$2^{\aleph_0}$ ends

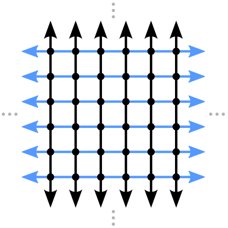
Theorem (Stallings).

TFAE for every group Γ with finite generating set S:

- Cay(Γ, S) has $\geqslant 2$ ends; [this is independent of S]
- Γ decomposes as a non-trivial amalgamated free product or HNN-extension over a finite subgroup.

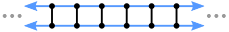
$\mathsf{Cay}(\Gamma, S)$ has $\geqslant 2$ ends

$\Leftrightarrow$ Γ decomposes as a non-trivial amalgamated free product or HNN-extension over a finite subgroup.



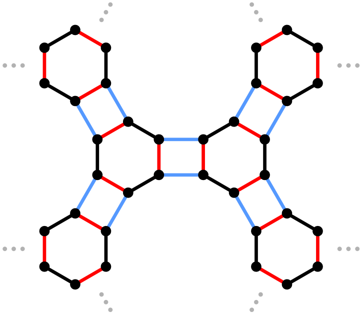
1 end

$\mathbb{Z} \times \mathbb{Z}$



2 ends

$\mathbb{Z}_2 \times \mathbb{Z}$ (HNN)



$2^{\aleph_0}$ ends

$D_3 *_{D_1} D_2$

Open problem. Extend Stallings' theorem to finite groups.

Challenges.

1. Ends have no finite counterparts
2. Key step of the proof fails for finite Γ

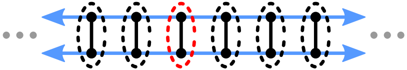
Hard: Cay(Γ, S) has $\geqslant 2$ ends $\Longrightarrow \Gamma$ decomposes

Hard: Cay(Γ, S) has $\geqslant 2$ ends $\Longrightarrow \Gamma$ decomposes

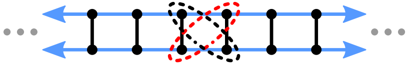
---

Proof idea. First, find separator X of Cay(Γ, S) that crosses no separators in its Γ-orbit:

X works



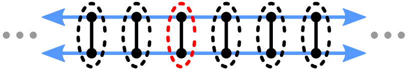
X crosses some $g \cdot X$



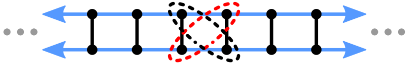
Hard: $\mathrm{Cay}(\Gamma, S)$ has $\geqslant 2$ ends $\Longrightarrow \Gamma$ decomposes

Proof idea. First, find separator X of $\mathrm{Cay}(\Gamma, S)$ that crosses no separators in its Γ-orbit:

X works



X crosses some $g \cdot X$



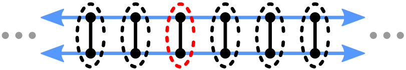
Then show that Γ decomposes over the stabilizer of X. $(\square)$

Hard: Cay(Γ, S) has $\geqslant 2$ ends $\Longrightarrow \Gamma$ decomposes

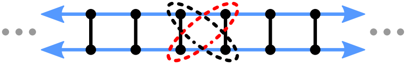
---

Proof idea. First, find separator X of Cay(Γ, S) that crosses no separators in its Γ-orbit:

X works

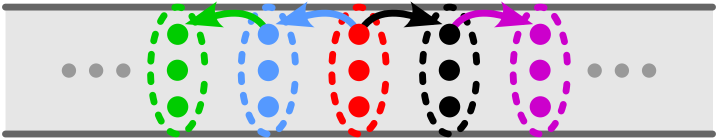


X crosses some $g \cdot X$



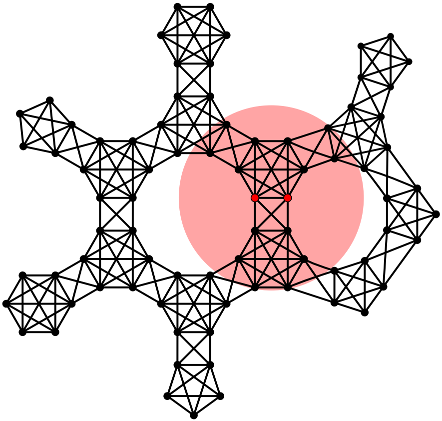
Then show that Γ decomposes over the stabilizer of X. $(\square)$

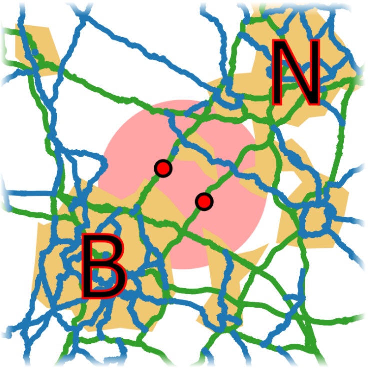
Challenge for finite Γ. If X exists, then Γ must be infinite:



Recent development in graph-minor theory:

local separators

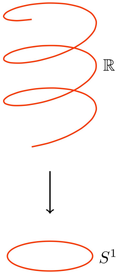




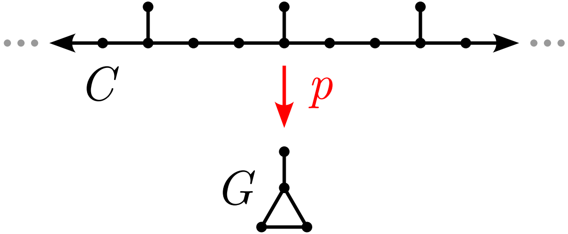
Rough idea: vertex-sets that separate G locally in a ball of given radius $r/2 > 0$, not necessarily G itself

A *covering* of G is a surjective graph-homomorphism $p\colon C \to G$ such that for every vertex $v \in C$:

- p restricts to a bijection $E_C(v) \to E_G(p(v))$.

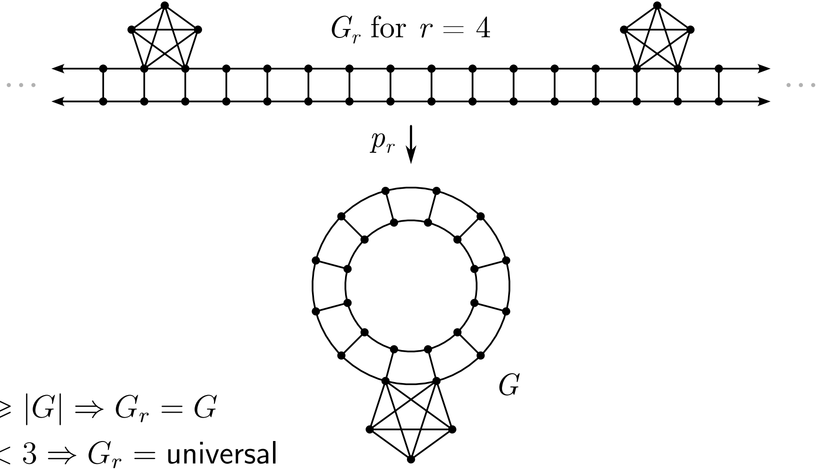


Example. Universal coverings are trees.



$\forall G$ and $r > 0$ there is a unique *r-local covering $p_r \colon G_r \to G$* s.t.

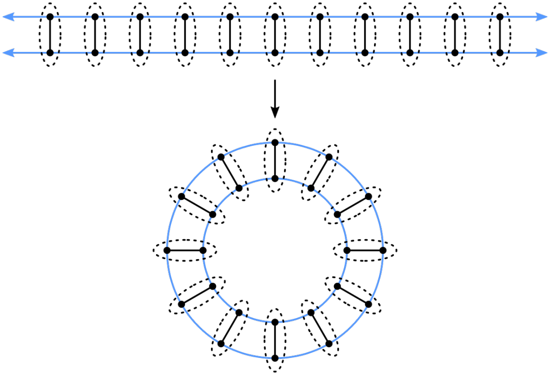
1. p_r restricts to an isomorphism $B_{G_r}(v, r/2) \to B_G(p_r(v), r/2)$ for every $v \in V(G_r)$, and
2. p_r is 'nearest' to the universal covering with (1).



$r \geqslant |G| \Rightarrow G_r = G$

$r < 3 \Rightarrow G_r =$ universal

r-local separators of G $:=$ projections of separators of G_r (roughly)



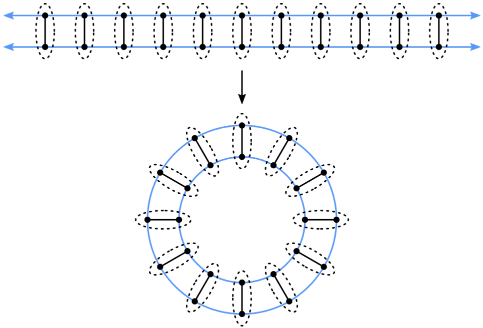
Ideas for challenges for finite Γ.

1. Use ends of r-local covering of some Cay(Γ, S).
2. Use Γ-orbit of suitable r-local separator in proof.

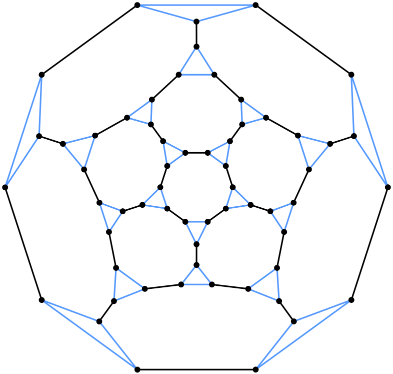
Main result (Carmesin, Kontogeorgiou, K., Turner '24)

TFAE for every finite nilpotent group Γ of class $\leqslant n$ and $r \geqslant 4^{n+1}$:

- The r-local covering of some Cayley graph G of Γ has $\geqslant 2$ ends that are separated by $\leqslant 2$ vertices.
- G has an r-local separator of size $\leqslant 2$ and $|\Gamma| > r$.
- $\Gamma \cong \mathbb{Z}_i \times \mathbb{Z}_j$ for some $i > r$ and $j \in \{1, 2\}$.



Why nilpotent?



r-local covering has $2^{\aleph_0}$ ends separated by cutvertices for $r \leqslant 9$.

But A_5 is simple.

Open problem. Extend main result to solvable groups.

Why only (local) separators of size $\leqslant 2$?

Heavily exploited in proof...

Theorem (Tutte 60s) Every 2-connected graph is either

- 3-connected,
- has a 2-separator that crosses no other 2-separator, or
- is a cycle.

Why only (local) separators of size $\leqslant 2$?

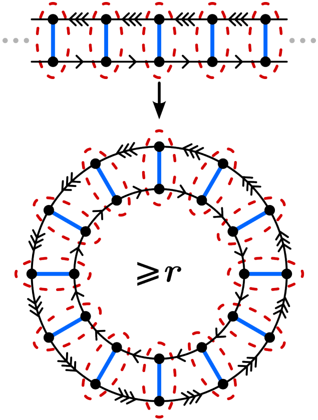
Heavily exploited in proof...

Theorem (Carmesin '20) Every r-**locally** 2-connected graph is

- r-**locally** 3-connected,
- has an r-**local** 2-separator that crosses no other r-**local** 2-separators, or
- is a cycle **of length** $\leqslant r$.

Open problem. Extend main result to local separators of size > 2.

Why are dihedral groups missing?



Answer. $r \geqslant 4^{n+1}$ is too bad.

Open problem. Improve bound on r. Dihedral groups should appear. What other new products appear?

Main result.

TFAE for every finite nilpotent group Γ of class $\leqslant n$ and $r \geqslant 4^{n+1}$:

- The r-local covering of some Cayley graph G of Γ has $\geqslant 2$ ends that are separated by $\leqslant 2$ vertices.
- G has an r-local separator of size $\leqslant 2$ and $|\Gamma| > r$.
- $\Gamma \cong \mathbb{Z}_i \times \mathbb{Z}_j$ for some $i > r$ and $j \in \{1, 2\}$.

Open: Solvable groups • Local (> 2)-separators • Better bounds.

arXiv:2403.07776 $\longrightarrow$

Slides: jan-kurkofka.eu

Thank you!