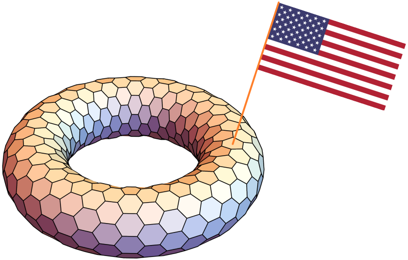
# Canonical decompositions of 3-connected graphs

Jan Kurkofka



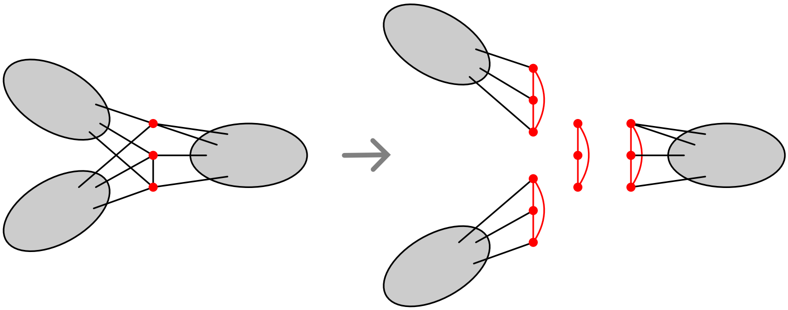
Joint work with Johannes Carmesin

University of Birmingham

FOCS 2023

Problem: Decompose $k$-con'd $G$ along $k$-separators
into pieces that are $(k+1)$-con'd or 'basic'.

*Decomposing* $G$ along a $k$-separator:



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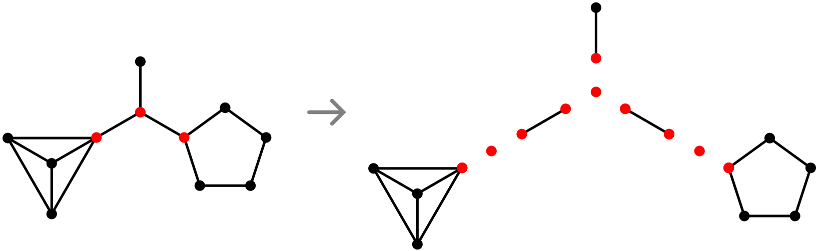

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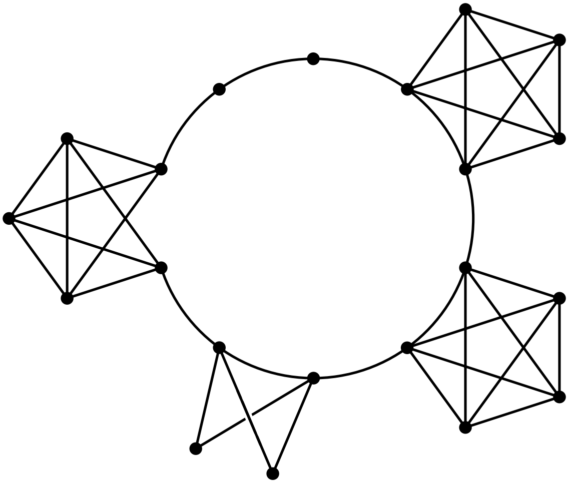

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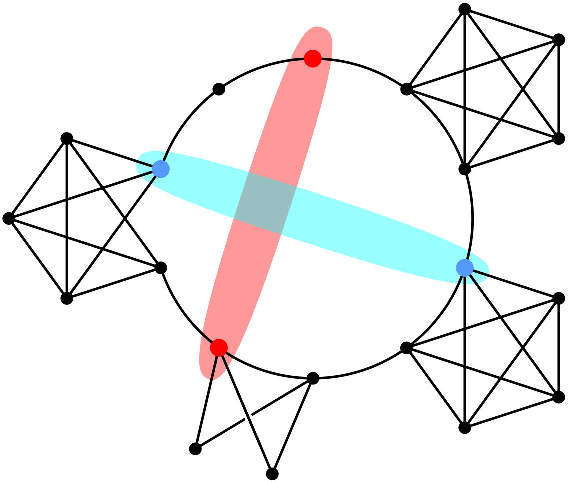

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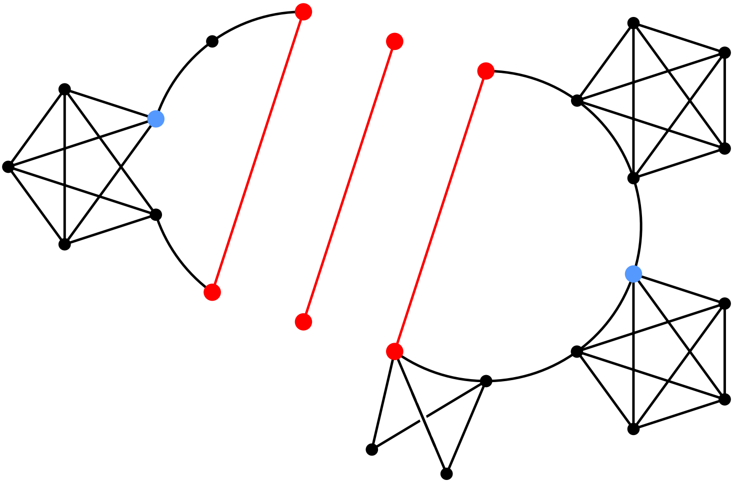
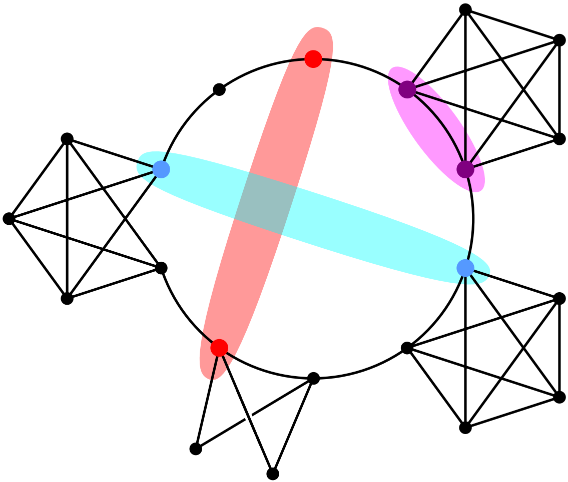


Two $k$-separators *cross* if they separate each other;

otherwise they are *nested*.



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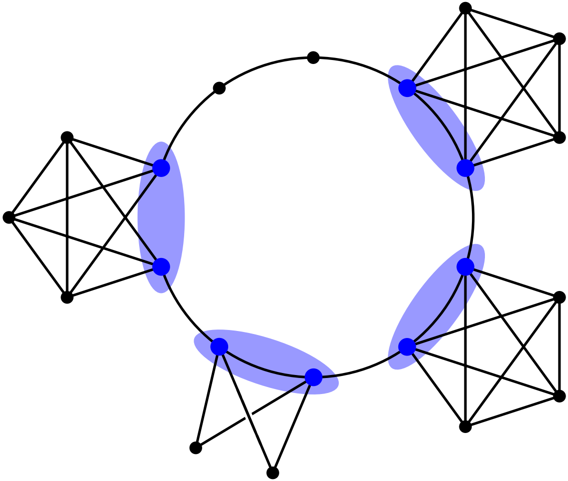
A $k$-separator is *totally-nested* if it is nested with every $k$-separator.



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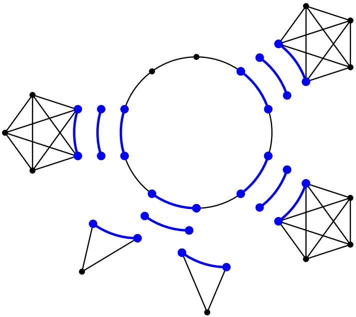
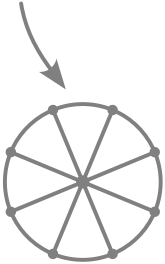


Theorem (Tutte 66), SPQR-trees

Every 2-con'd $G$ decomposes along its totally-nested 2-separators into 3-con'd graphs, cycles and $K_2$'s.

Guess

Every 3-con'd $G$ decomposes along its totally-nested 3-separators into 4-con'd graphs, wheels and $K_3$'s.



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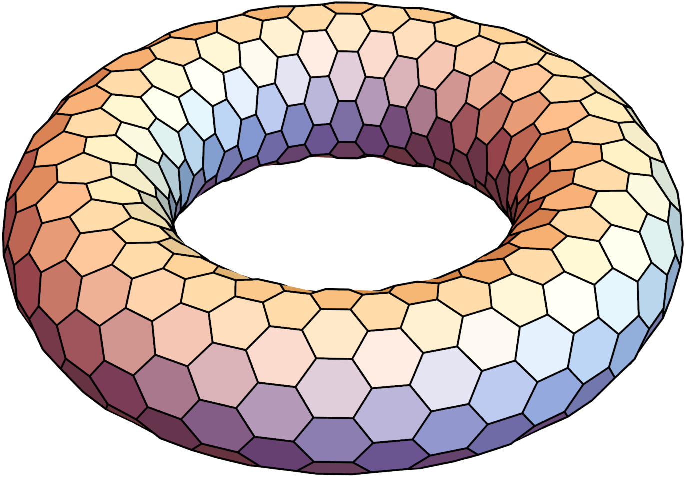

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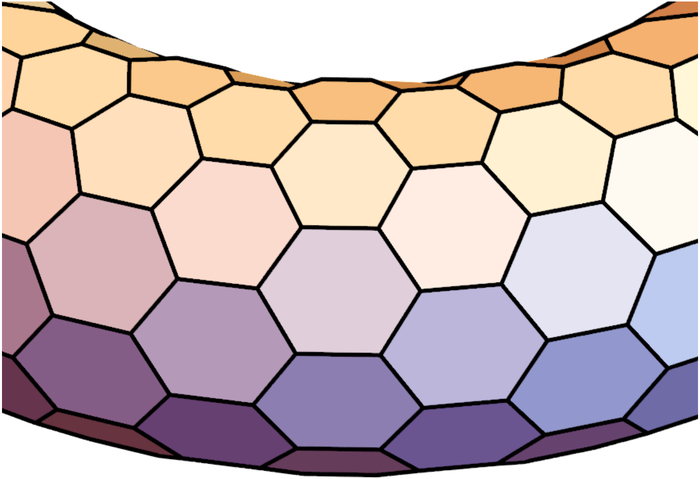

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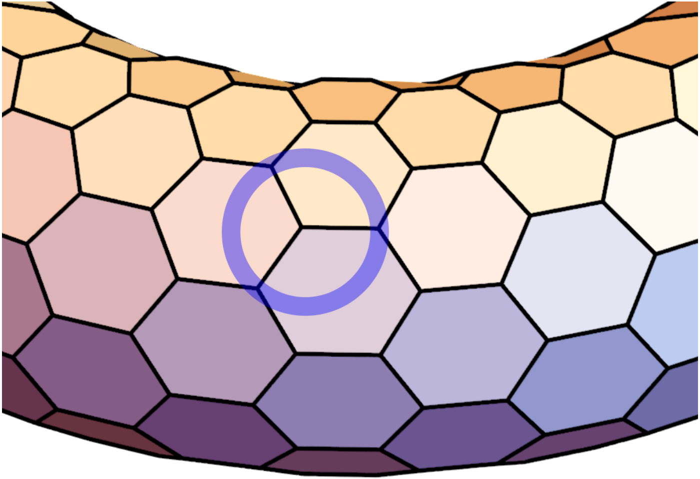

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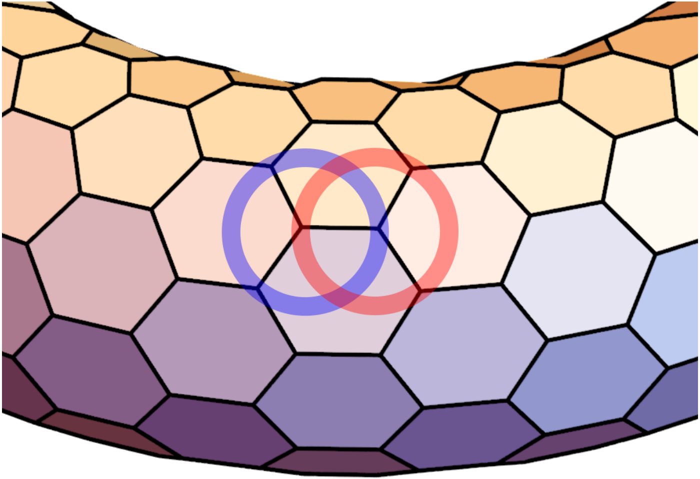
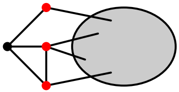
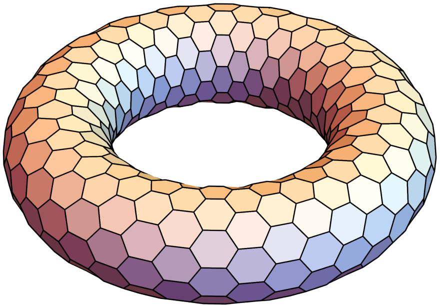


Guess

Every 3-con'd $G$ decomposes along its totally-nested 3-separators into quasi 4-con'd graphs, wheels and $K_3$'s.

↓

3-con'd, $> 4$ vertices, every 3-separator has form 



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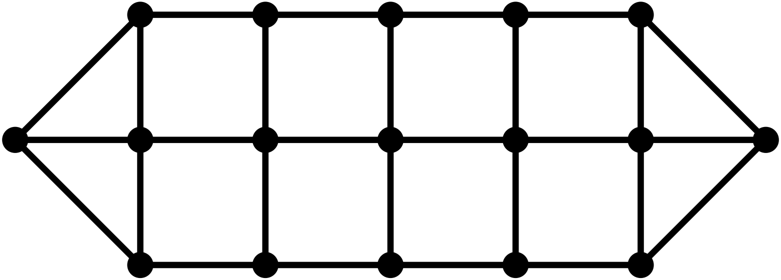

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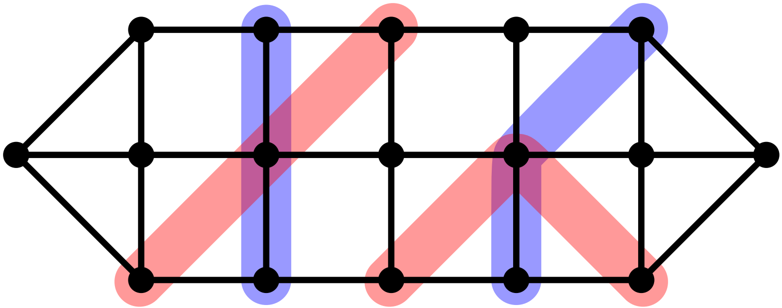

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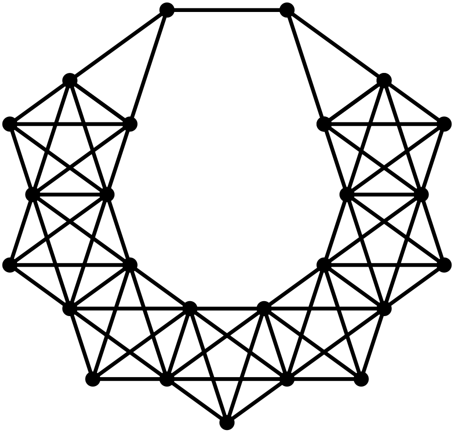

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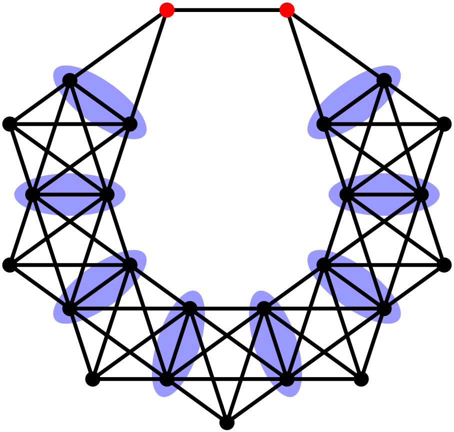

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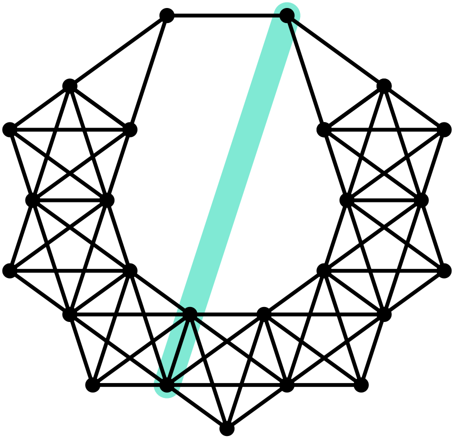

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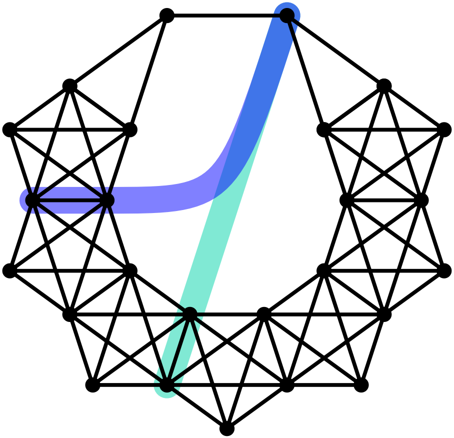

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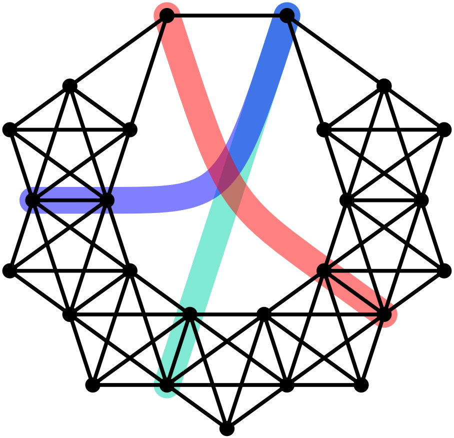

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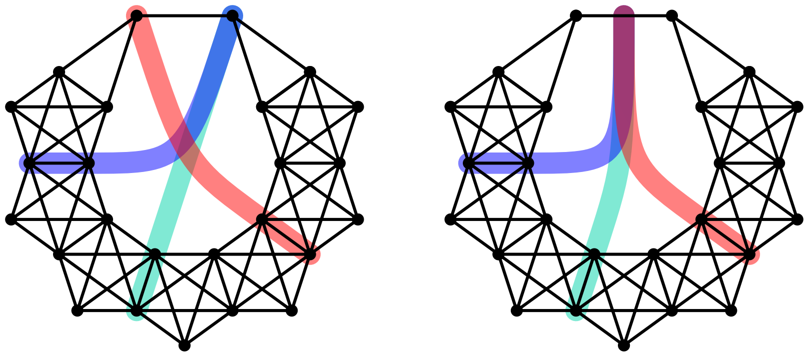

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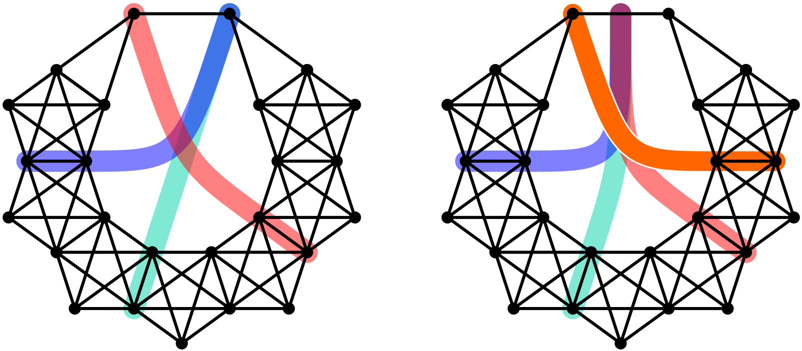

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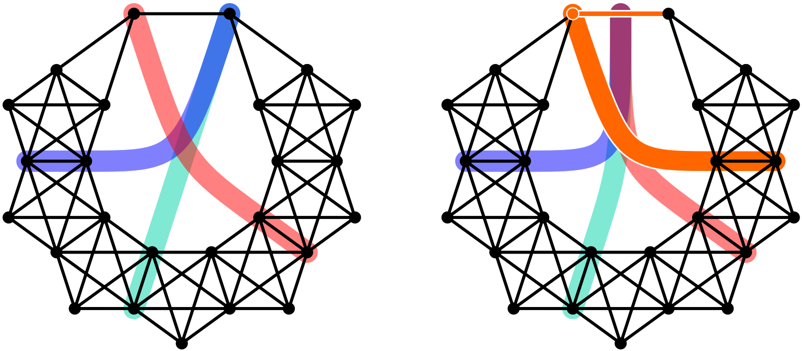

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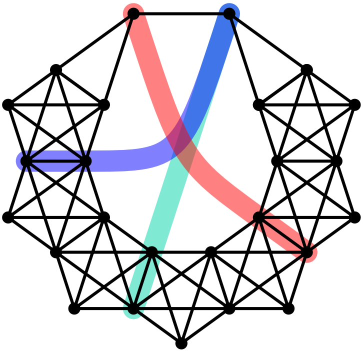
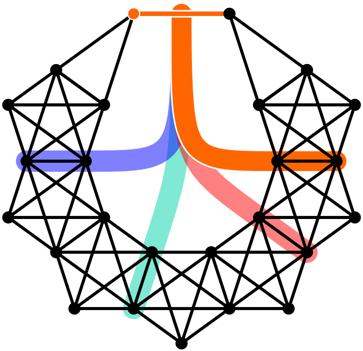
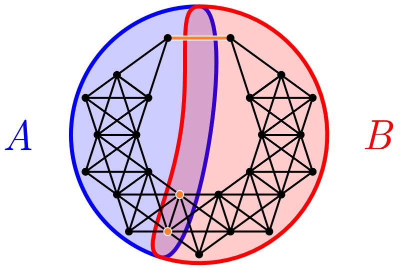


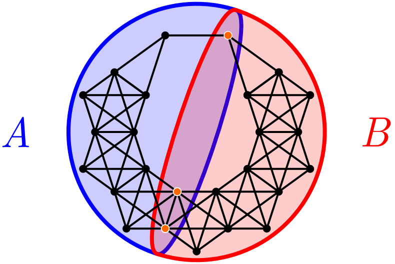





*mixed-separation* of $G$: $\{A, B\}$ with $A \cup B = V(G)$ and both $A \setminus B$ and $B \setminus A$ nonempty

*separator* of $\{A, B\}$: $(A \cap B) \cup E(A \setminus B, B \setminus A)$

*tri-separation* of $G$: mixed-sep'n $\{A, B\}$ with $|\text{sep'r}| = 3$ and every vx in $A \cap B$ has two neighb's in $G[A]$ and in $G[B]$

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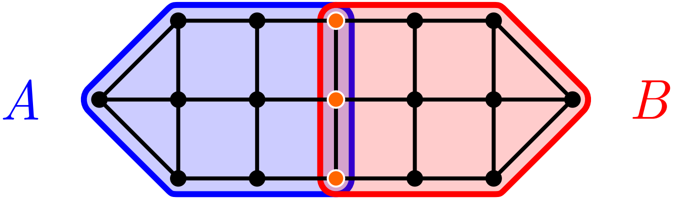


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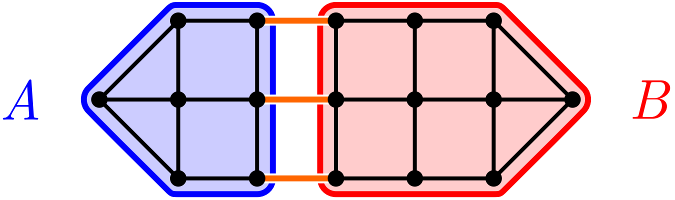


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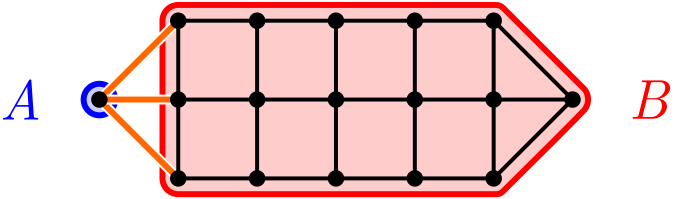
trivial

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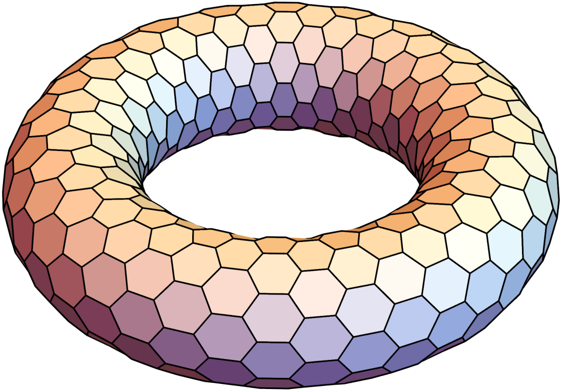
totally-nested nontrivial tri-separations



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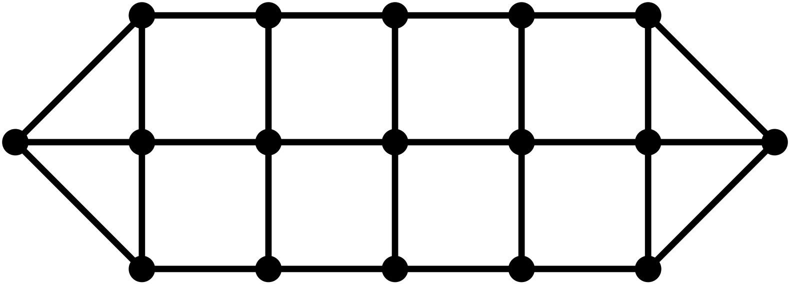

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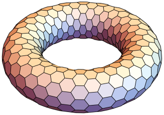

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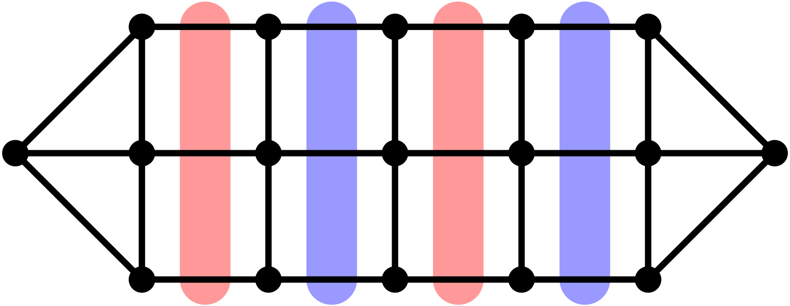

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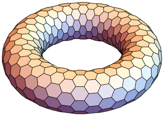

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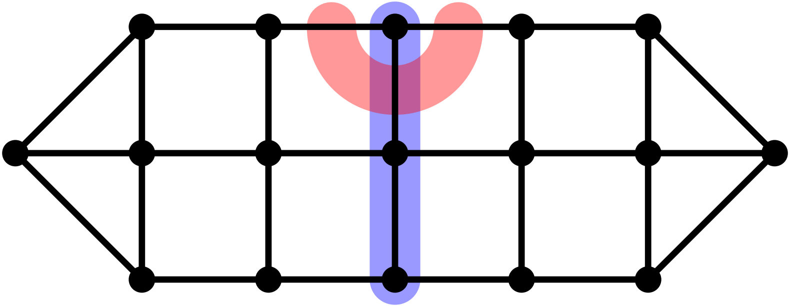

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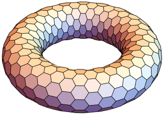

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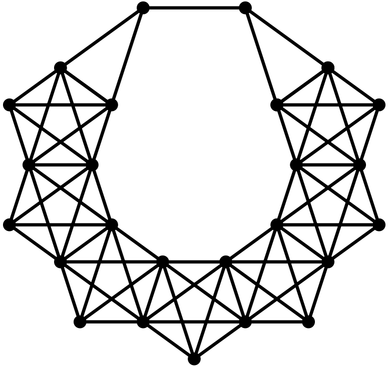

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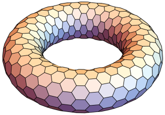
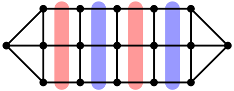
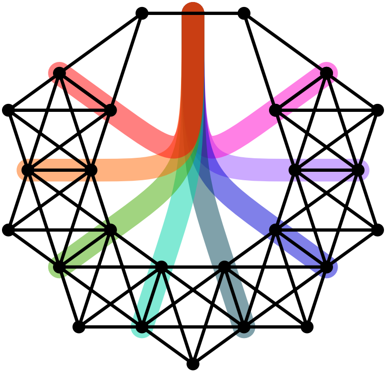




# totally-nested nontrivial tri-separations



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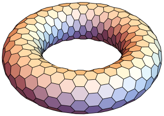
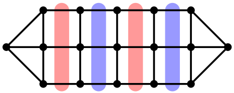
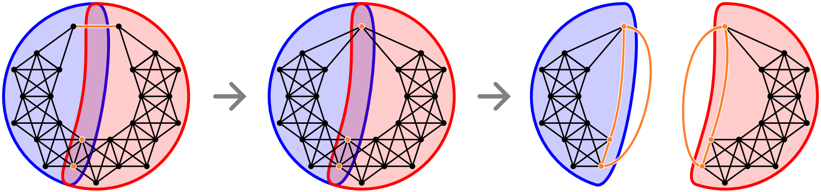
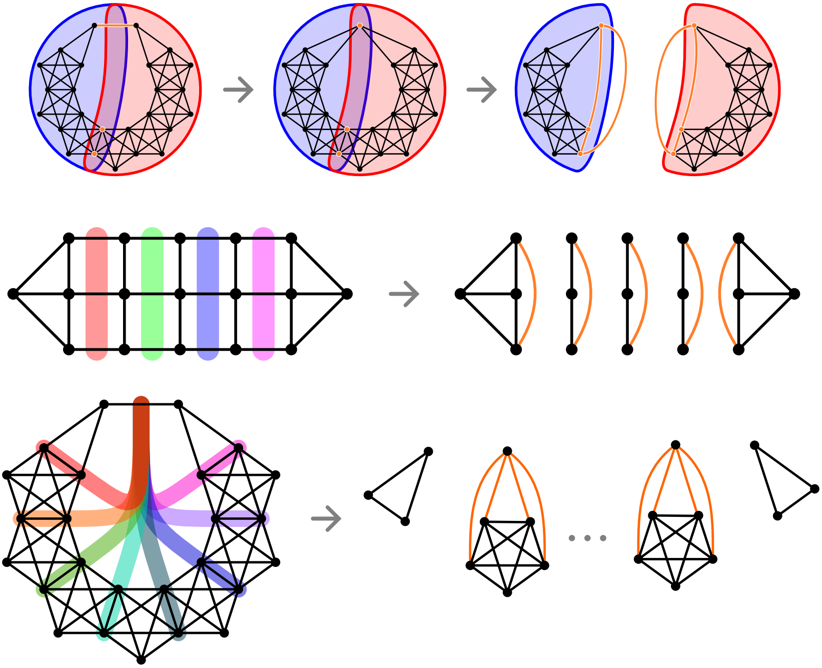




## Decomposing along a tri-separation



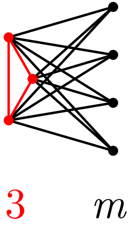


Main result (Carmesin & K. 23)

Every 3-con'd $G$ decomposes along its totally-nested nontrivial tri-separations into minors of $G$ that are

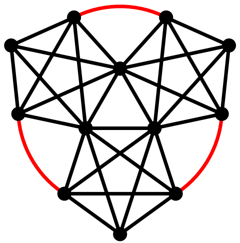
- quasi 4-con'd
- wheels
- thickened $K_{3,m}$

or $G = K_{3,m}$ ($m \geqslant 0$).



| | Grohe 16 | Carmesin & K. 23 |
|---|---|---|
| method | recursive | Tutte (totally nested) |
| decomposition | 3-separations | tri-separations |
| torsos | $K_4$, | wheels, |
| | quasi 4-con'd, $K_3$ | quasi 4-con'd, thickened $K_{3,m}$ |
| canonical | no | yes |
| algorithm | $O(n^2(n+m))$ | ??? |

## Application: Connectivity Augmentation from 0 to 4



Theorem (Carmesin & Sridharan 23+)

$\exists$ FPT-algorithm with runtime $C(\ell) \cdot \mathrm{Poly}(\,|V(G)|\,)$ and

Input: Graph $G$, $\ell \in \mathbb{N}$ and $F \subseteq E(\,\overline{G}\,)$

Output: No, or $\leqslant \ell$-sized $X \subseteq F$ such that $G + X$ is 4-con'd

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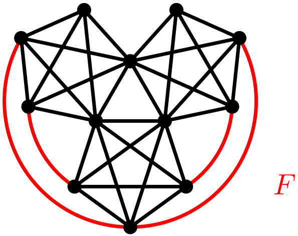


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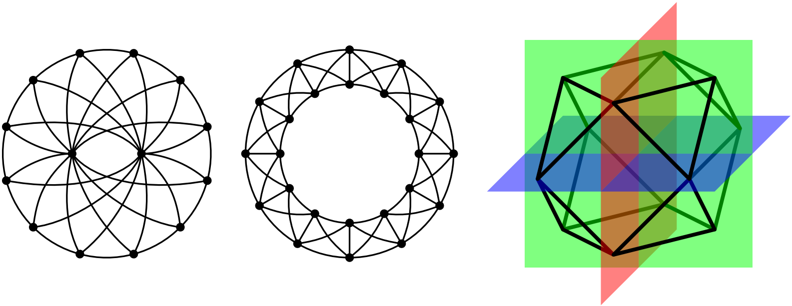
Open: Extend the main result to $k$-separations for $k \geqslant 4$.



Open: Efficient algorithms?

Open: Directed graphs?

$k = 1$: Bowler, Gut, Hatzel, Kawarabayashi, Muzi, Reich 23

$k \geqslant 2$: ???

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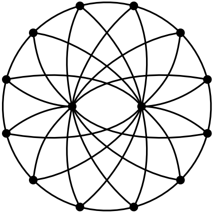

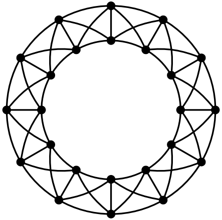

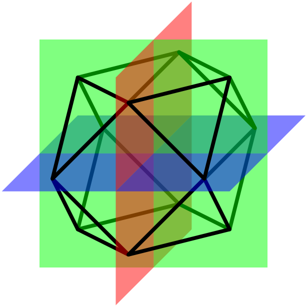

Open: Efficient algorithms?

Open: Directed graphs?

$k = 1$: Bowler, Gut, Hatzel, Kawarabayashi, Muzi, Reich 23

$k \geqslant 2$: ???

Thank you!