A Tutte-type canonical decomposition of 3- and 4-connected graphs

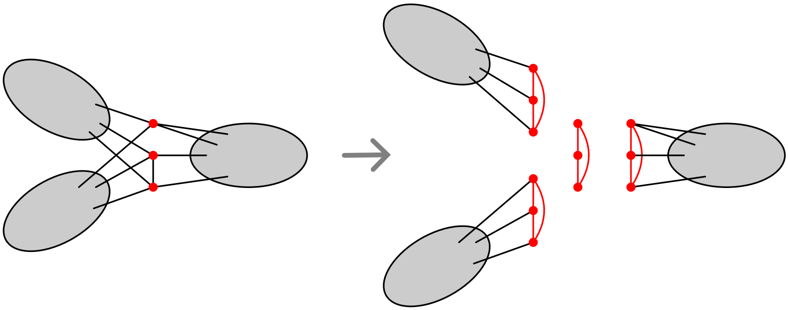
Jan Kurkofka

Joint work with Tim Planken

Ukrainian Summer School in Combinatorics 2025

Problem: Decompose k-con'd G along k-separators into parts that are $(k+1)$-con'd or 'basic'.

Decomposing G along a k-separator:



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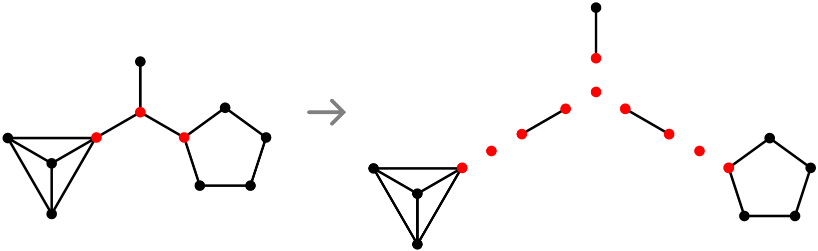

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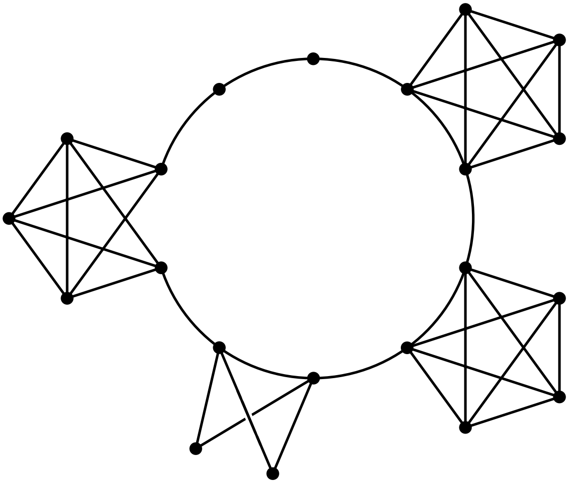

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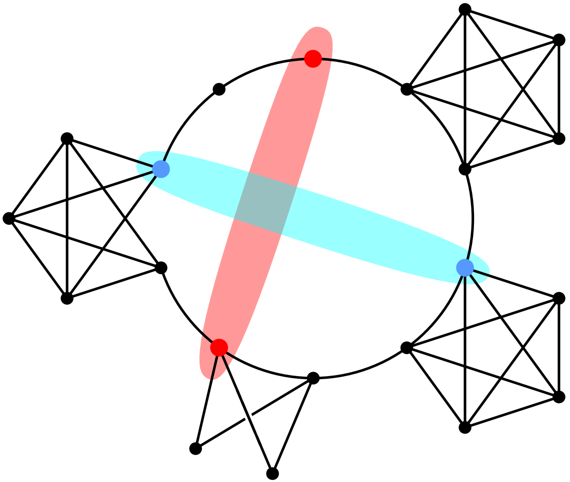

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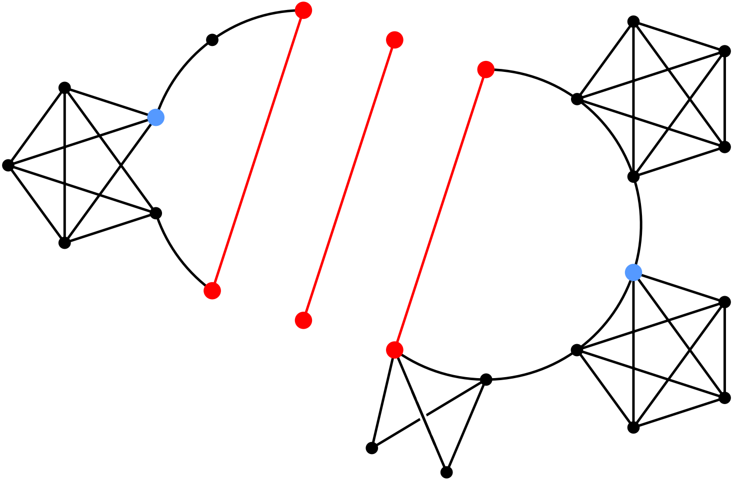
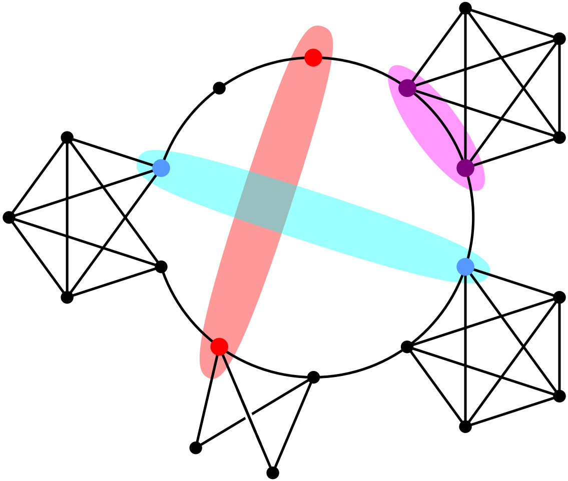


Two k-separators *cross* if they separate each other;

otherwise they are *nested*.



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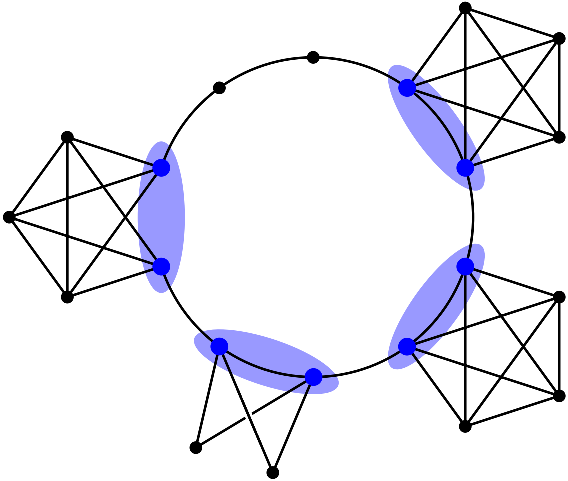

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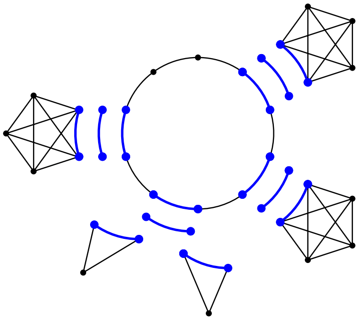
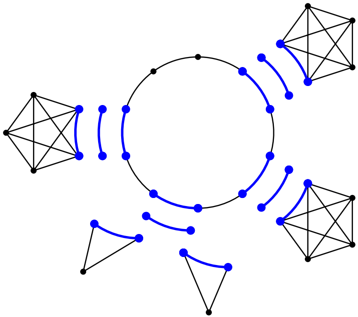


Theorem (Tutte 66), SPQR-trees

Every 2-con'd G decomposes along its totally-nested 2-separators into 3-con'd graphs, cycles and K_2's.

- **canonical**: isomorphisms map parts to parts
- **tree-decomposition** (for fans)



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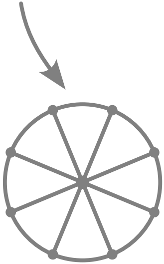
Guess

Every 3-con'd G decomposes along its totally-nested 3-separators into 4-con'd graphs, wheels and K_3's.



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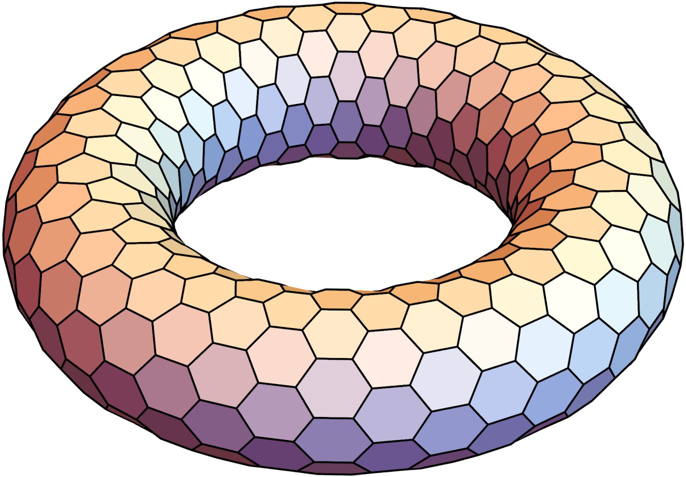

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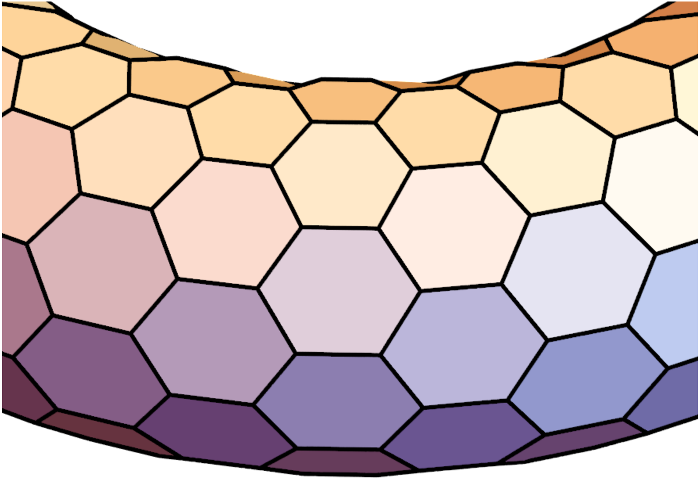

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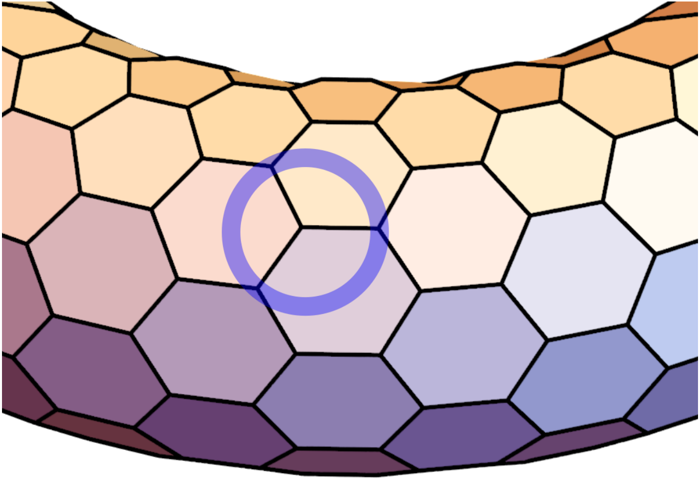

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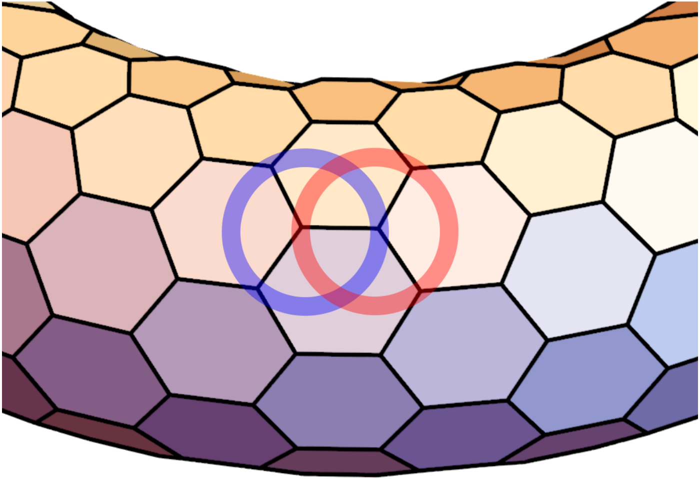
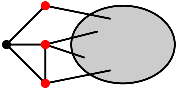
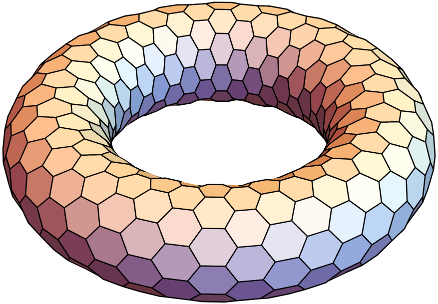


Guess

Every 3-con'd G decomposes along its totally-nested 3-separators into quasi 4-con'd graphs, wheels and K_3's.

↓

3-con'd and every 3-separator has form 



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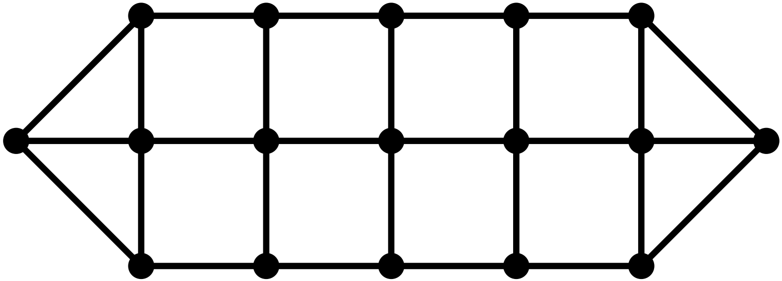

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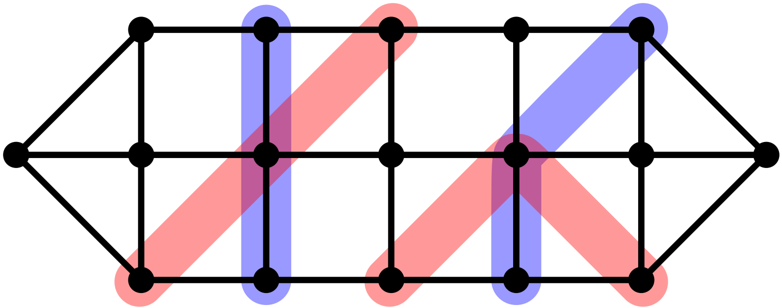

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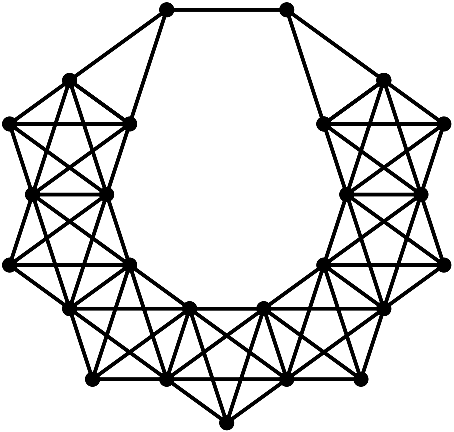

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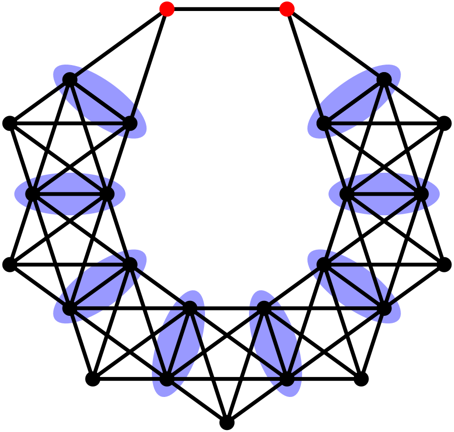

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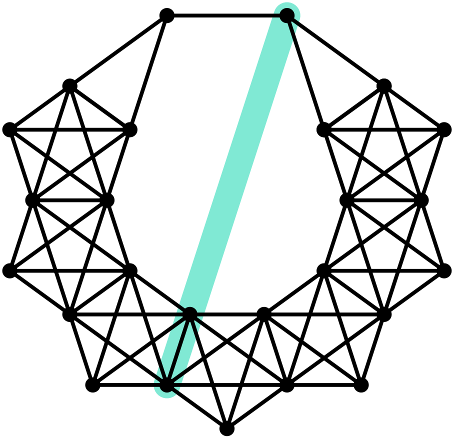

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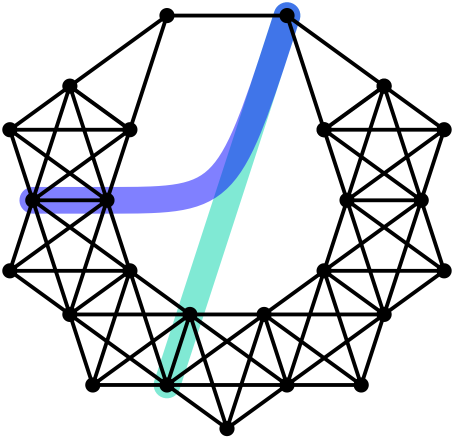

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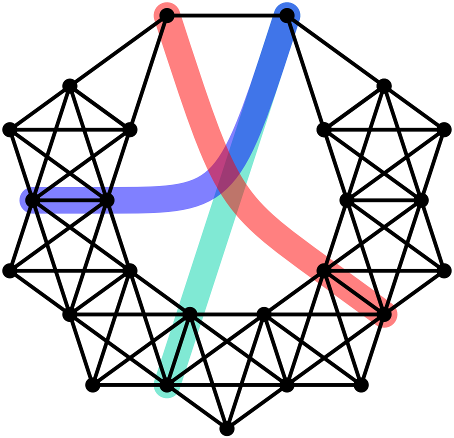

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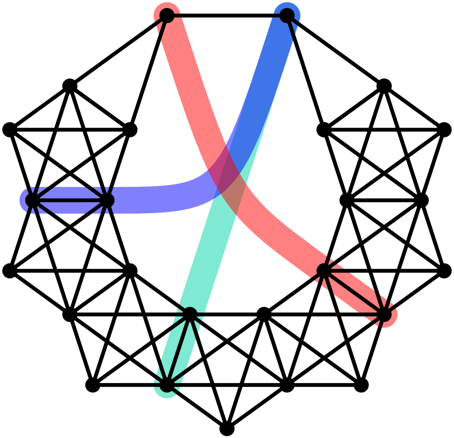

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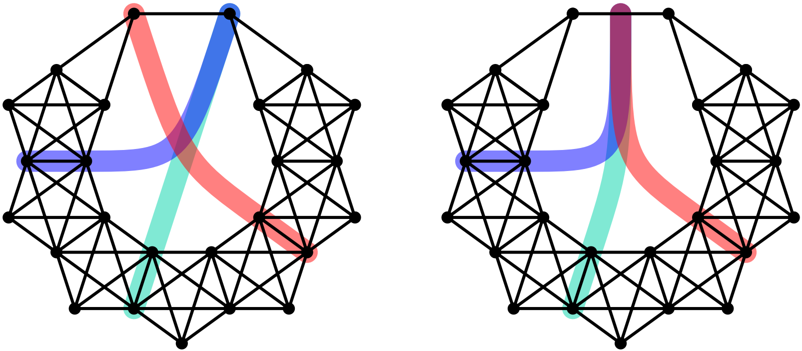

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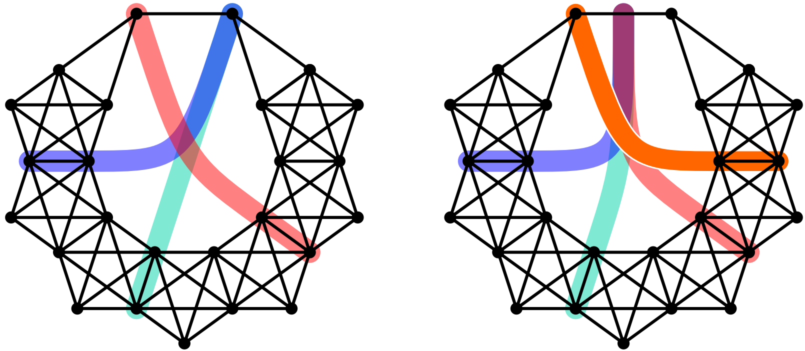

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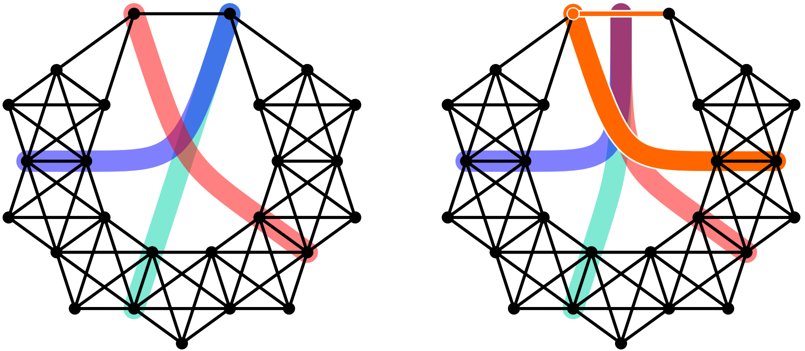

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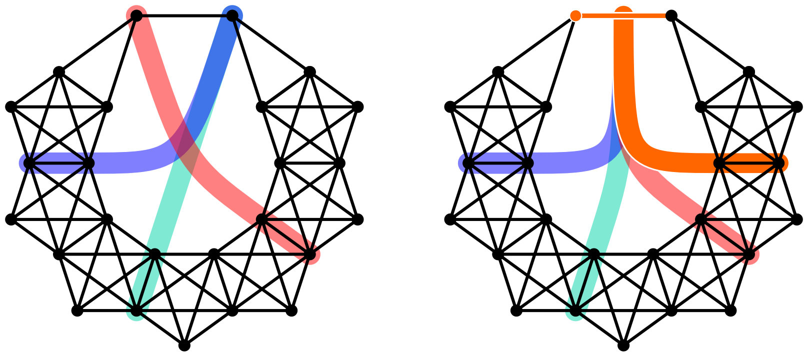
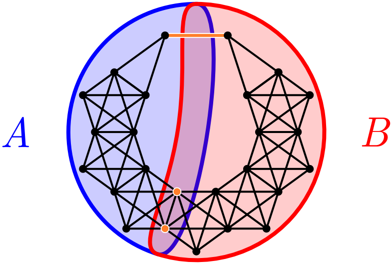
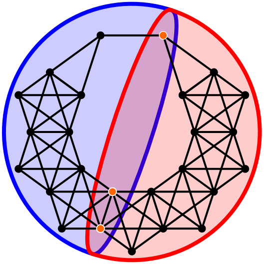




mixed-separation of G: (A, B) with $A \cup B = V(G)$ and $A, B \neq V(G)$

separator of (A, B): $(A \cap B) \cup E(A \setminus B, B \setminus A)$

A **tri-separation** of G is a mixed-sep'n (A, B) with $|\text{sep'r}| = 3$ s.t. every vx in $A \cap B$ has $\geqslant 2$ neighb's in A and B.



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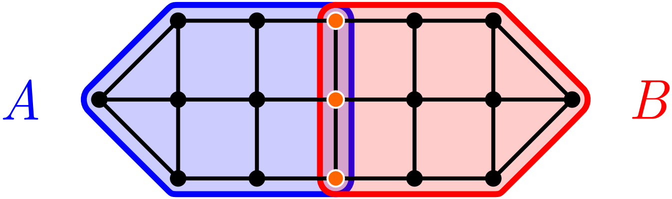

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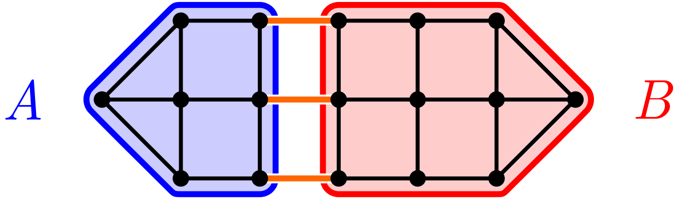

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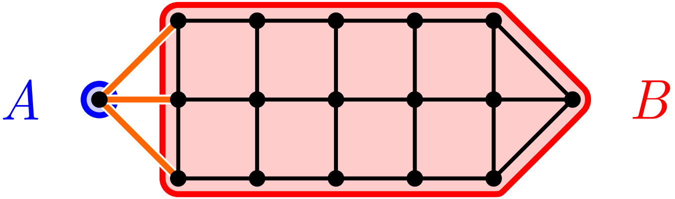
trivial



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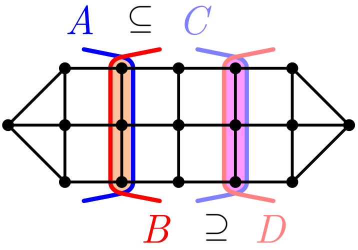
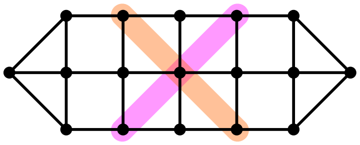
(A, B) and (C, D) are *nested* if $A \subseteq C$ and $B \supseteq D$ after possibly switching A with B or C with D; otherwise they *cross*.

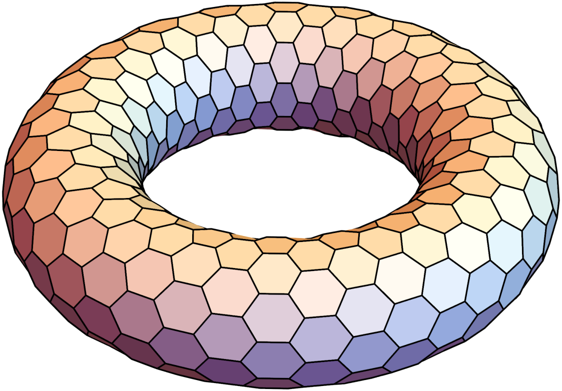
nested



crossing



totally-nested nontrivial tri-separations



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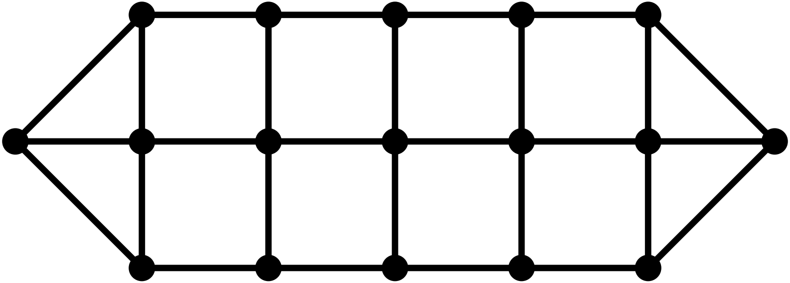

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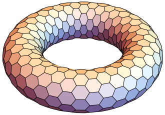

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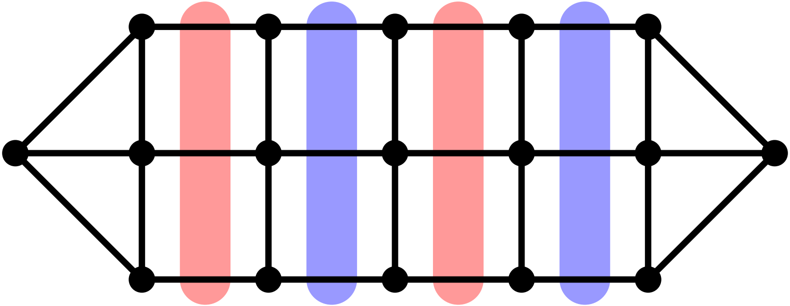

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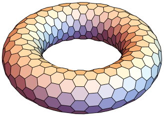

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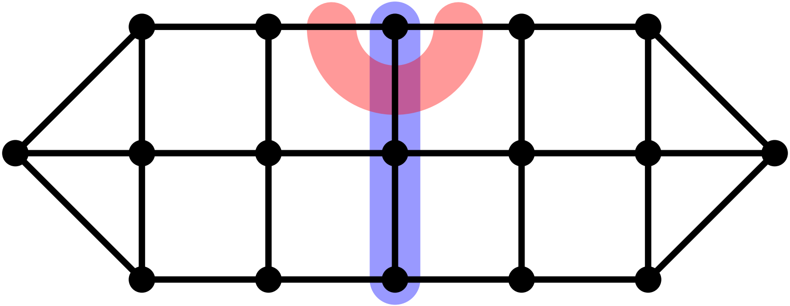

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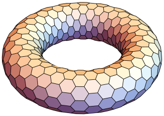

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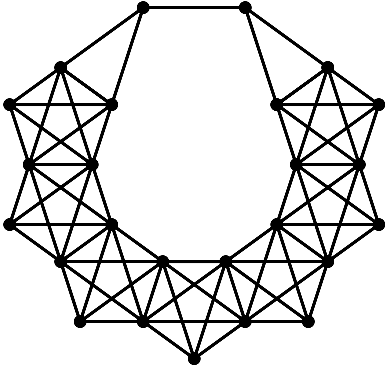

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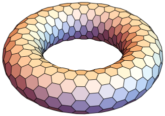

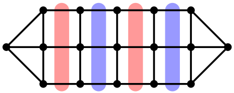

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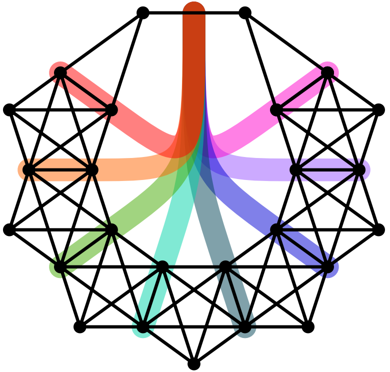

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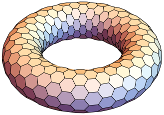
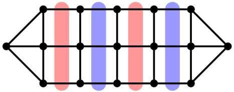
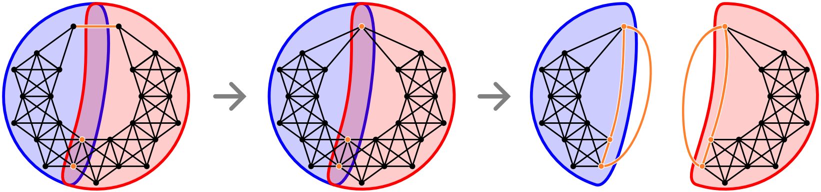
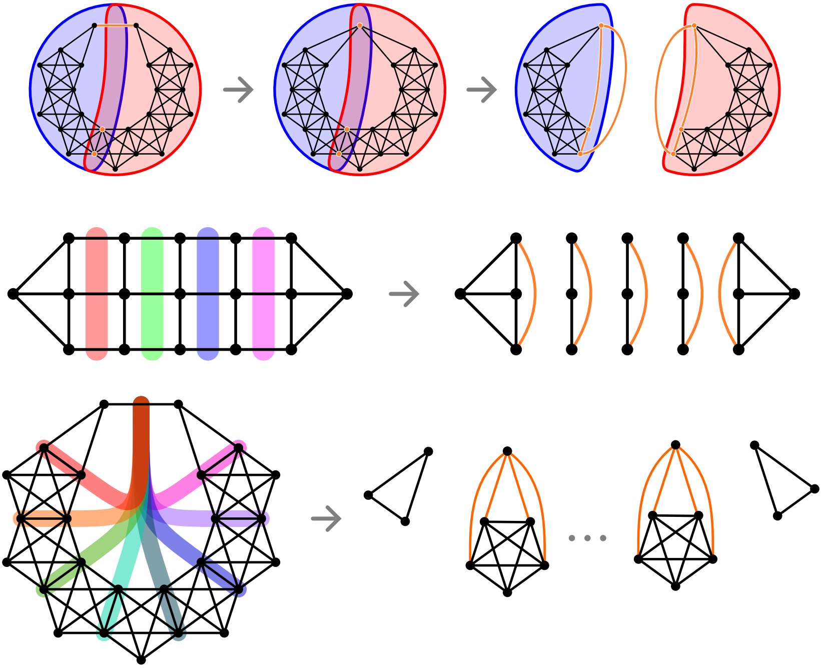




Decomposing along a tri-separation

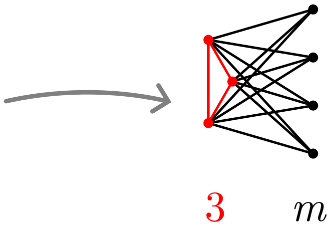




Theorem (Carmesin & K. 23)

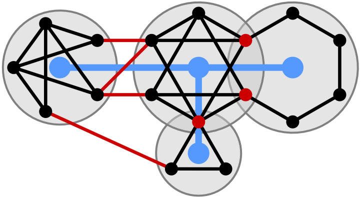
Every 3-con'd G decomposes along its totally-nested nontrivial tri-separations into parts that are

- quasi 4-con'd
- wheels
- thickened $K_{3,m}$

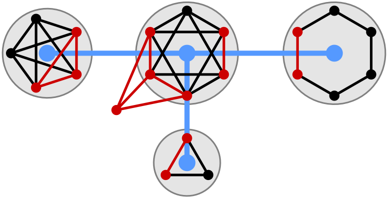


- canonical ✓
- for tree-decomposition fans:

mixed-tree-decomposition

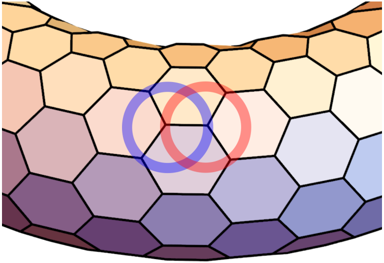


torsos



$k \geqslant 4$?

Challenge 1: Torus-example for $k \geqslant 3$



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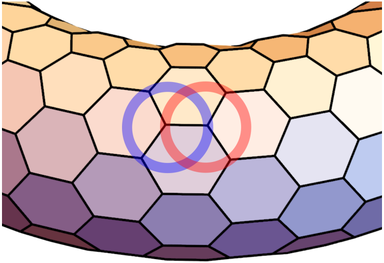

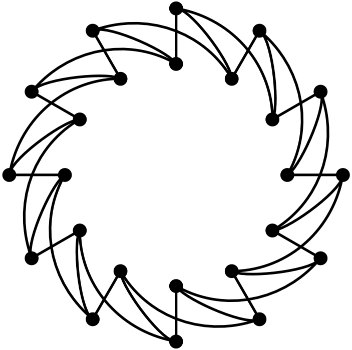

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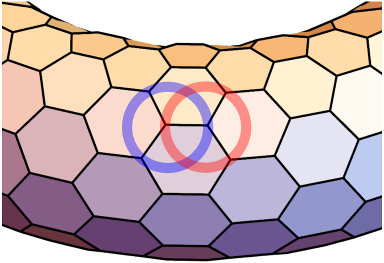
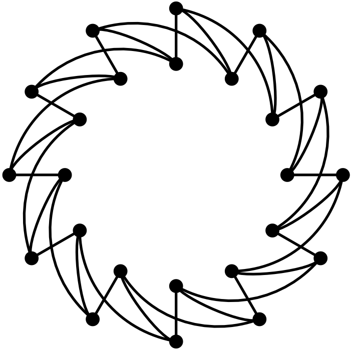
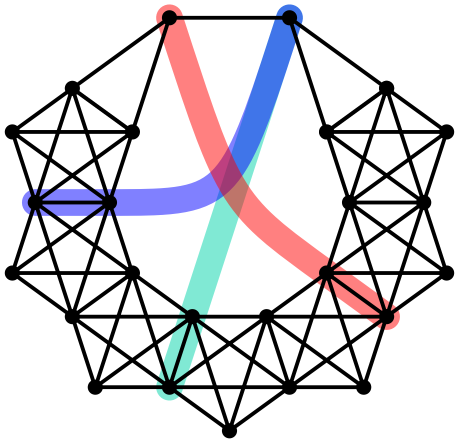
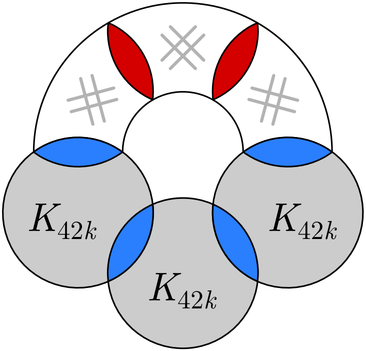




quasi-k-connected: k-con'd and every k-sep'r cuts off $\leqslant 1$ vertex

Challenge 2



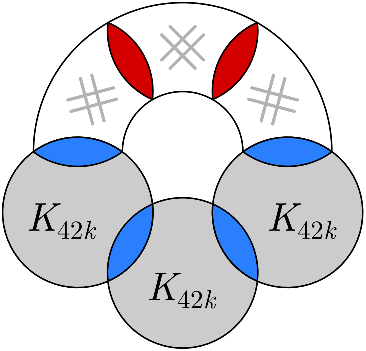


k odd

red: $\lfloor k/2 \rfloor$-cliques

blue: $\lceil k/2 \rceil$-cliques

Challenge 2



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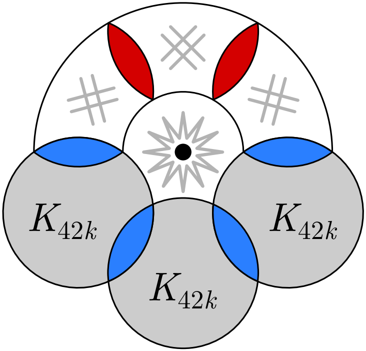
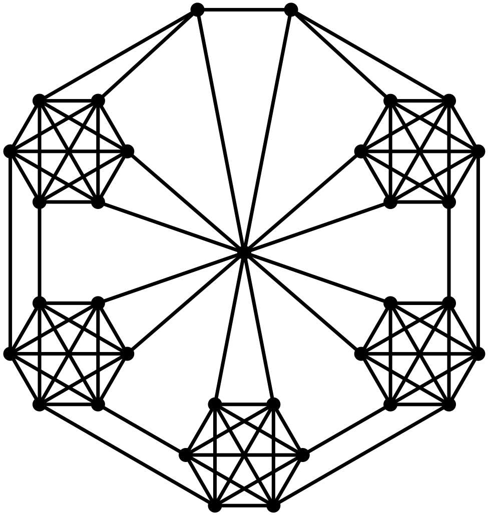


k even

red: $(\frac{k}{2} - 1)$-cliques

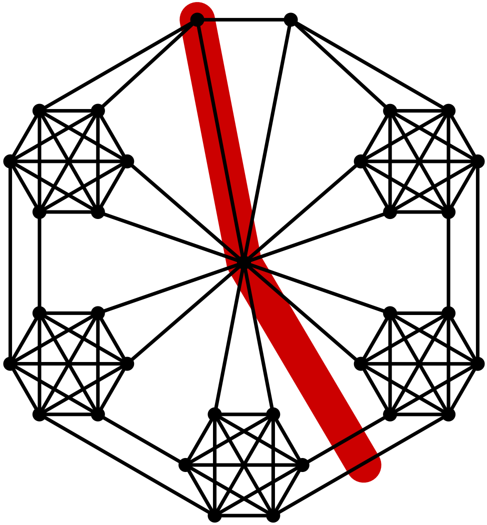
blue: $\frac{k}{2}$-cliques

Challenge 2



Challenge 2

Verbatim extension of tri-sep'ns to $k = 4$:



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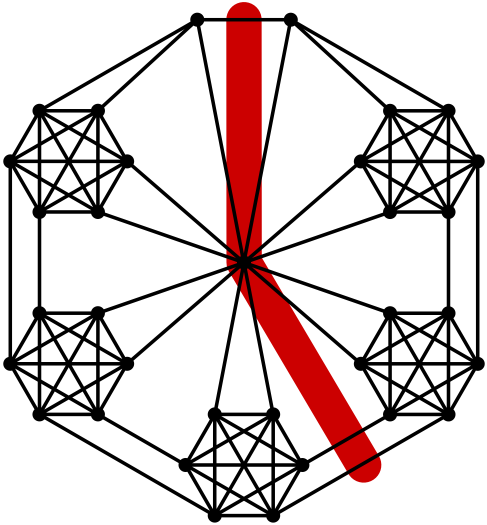

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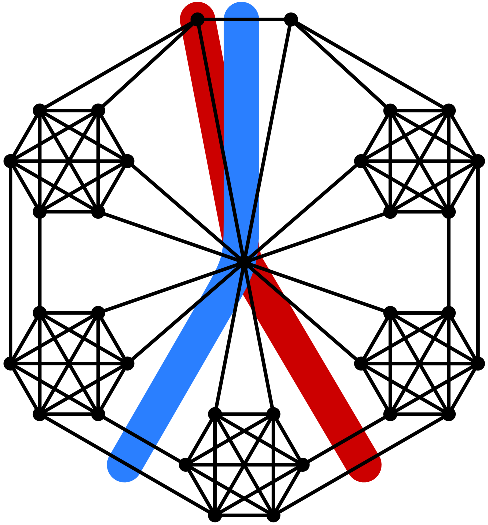
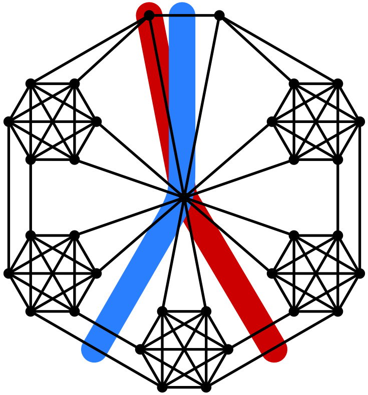
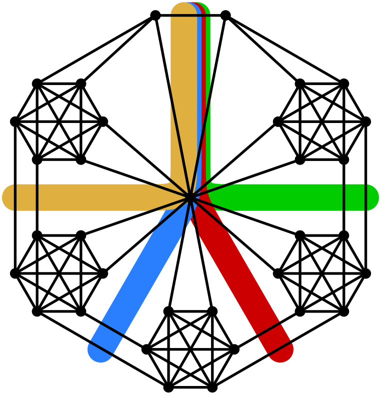




A **tetra-separation** is a mixed-sep'n (A, B) with |sep'r| $= 4$ s.t.:

- every vx in $A \cap B$ has $\geqslant 2$ neighb's in $A \setminus B$ and $B \setminus A$
- the edges in the sep'r form a matching



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Main result (K. & Planken 25)

Every 4-con'd G decomposes along its totally-nested tetra-separations into parts that are

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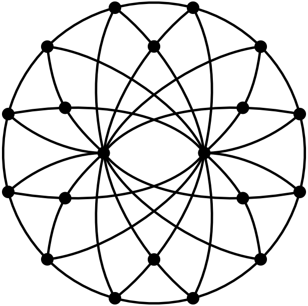

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Every 4-con'd G decomposes along its totally-nested tetra-separations into parts that are

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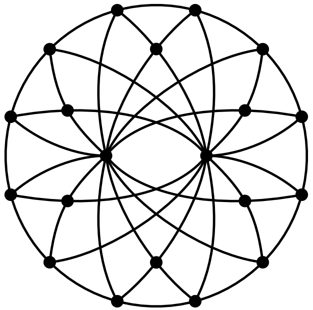

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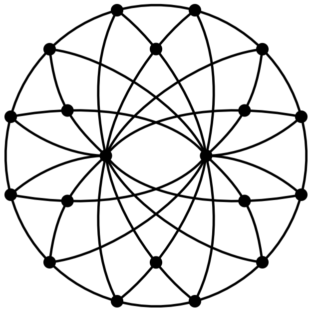
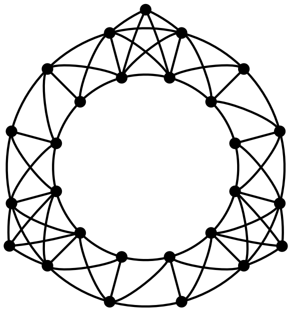
- cycle of triangles and 3-con'd graphs on $\leqslant 5$ vxs.





con'd

con'd

block-cut ⇓

	con'd	
block-cut	$\Downarrow$	
	2-con'd	K_2, K_1

	con'd	
block-cut	$\Downarrow$	
	2-con'd	K_2, K_1
Tutte	$\Downarrow$	

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	2-con'd	K_2, K_1
Tutte	$\Downarrow$	
	3-con'd	cycle, K_2

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block-cut	$\Downarrow$	
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tri-sep'ns	$\Downarrow$	

	con'd	
block-cut	$\Downarrow$	
	2-con'd	K_2, K_1
Tutte	$\Downarrow$	
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tri-sep'ns	$\Downarrow$	
	quasi-4-con'd	wheel, thickened K_3

	con'd	
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		con'd	
block-cut	$\Downarrow$		
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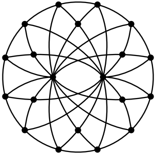
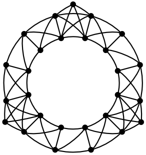
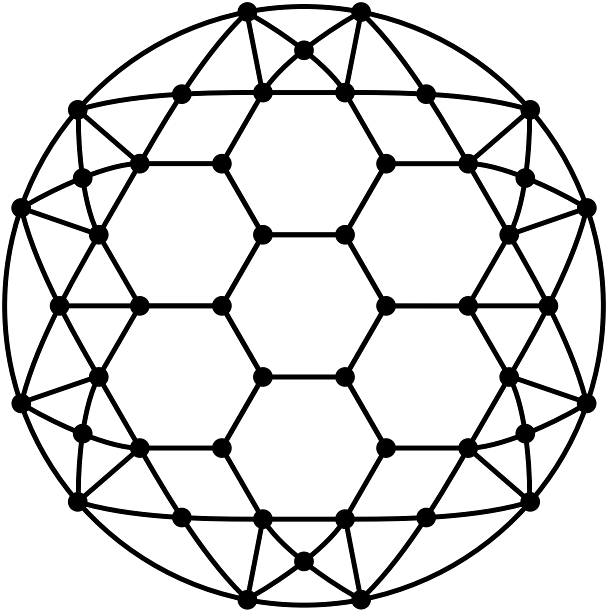
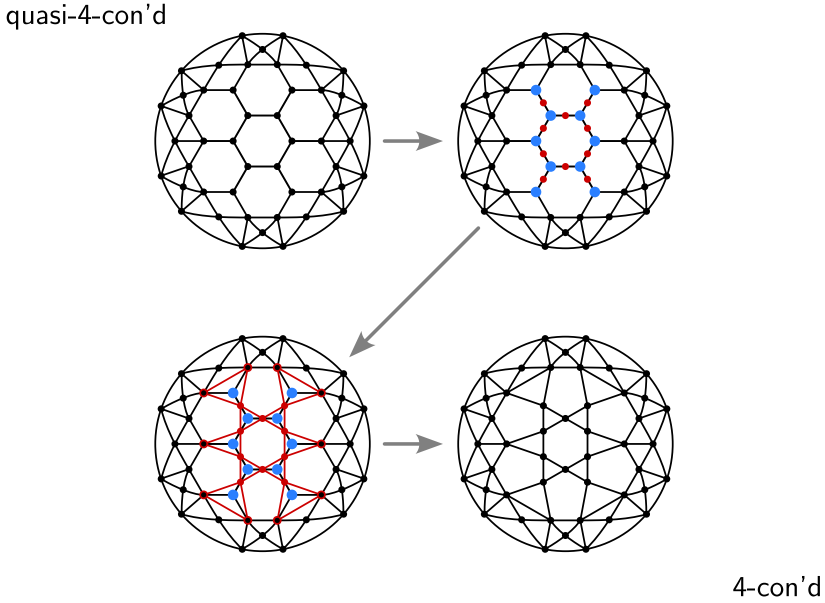
		quasi-5-con'd	  . . .



canonical Y–Δ transformation



	con'd	
block-cut	$\Downarrow$	
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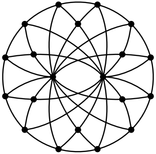
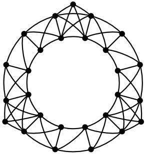
Tutte	$\Downarrow$	
	3-con'd	cycle, K_2
tri-sep'ns	$\Downarrow$	
	quasi-4-con'd	wheel, thickened K_3
Y–Δ	$\Downarrow$	
	4-con'd	
tetra-sep'ns	$\Downarrow$	
	quasi-5-con'd	  . . .

Problem: Classify all **vertex-transitive** finite con'd G

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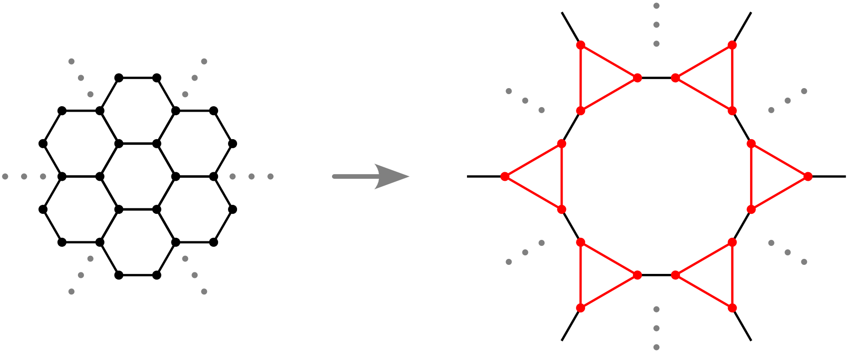
Approach: Low connectivity first

Theorem (Carmesin & K. 23)

Every **vertex-transitive** finite con'd G is either

- a cycle, K_2, K_1,
- quasi-4-con'd, or
- K_3-expansion of a quasi-4-con'd 3-regular arc-transitive graph.

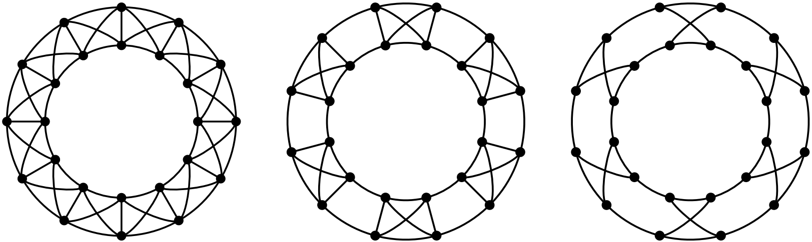


Theorem (K. & Planken 25)

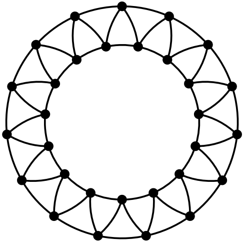
Every **quasi-4-con'd vertex-transitive** finite G is either

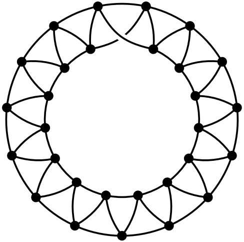
- bagel-like,
- cube-like,
- quasi-5-con'd, or
- K_4/C_4-expansion of quasi-5-con'd 4-regular arc-transitive graph.

bagel-like

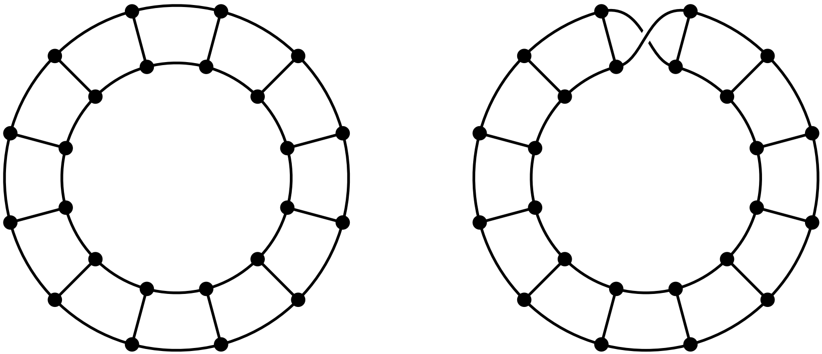


bagel-like

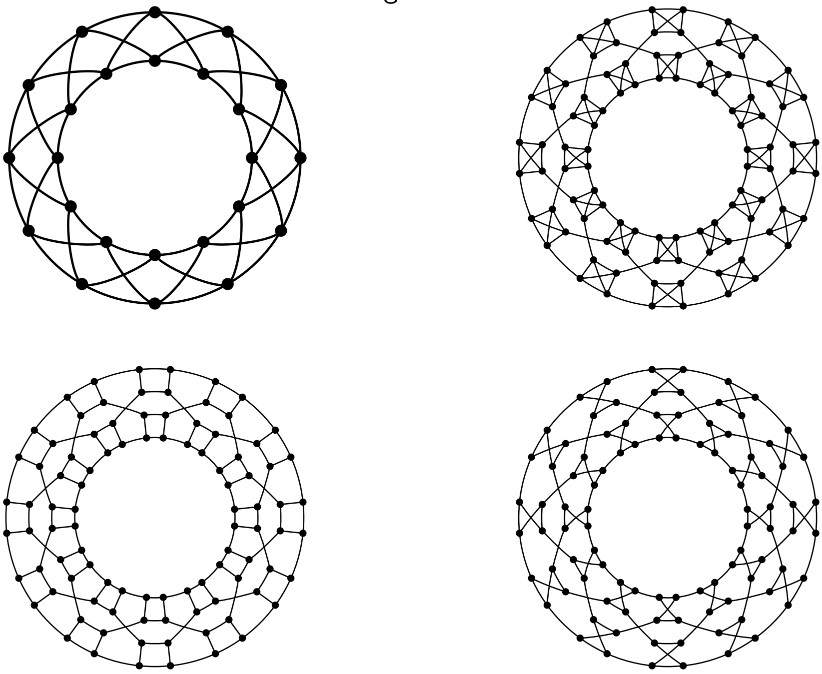




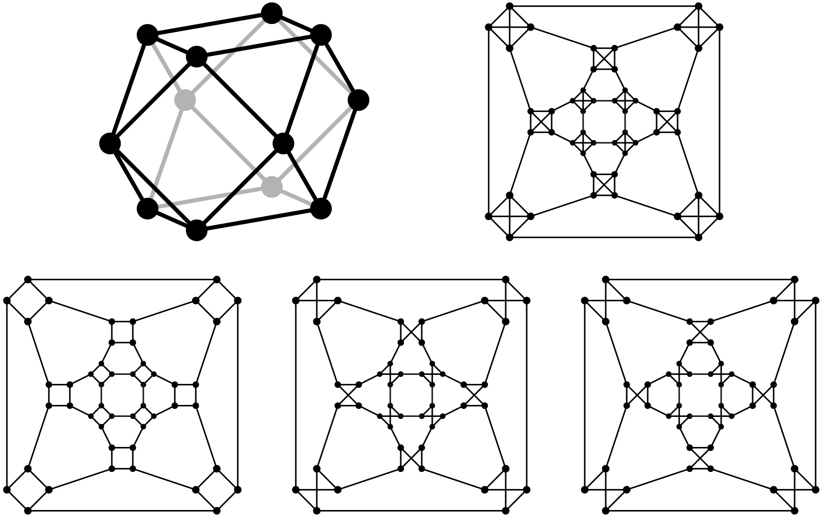
bagel-like



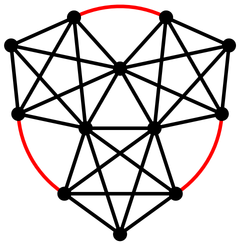
bagel-like



cube-like



Application: Connectivity Augmentation from 0 to 4



Theorem (Carmesin & Sridharan 25+)

$\exists$ FPT-algorithm with runtime $C(\ell) \cdot \mathrm{Poly}(\,|V(G)|\,)$ and

Input: Graph G, $\ell \in \mathbb{N}$ and $F \subseteq E(\,\overline{G}\,)$

Output: No, or $\leqslant \ell$-sized $X \subseteq F$ such that $G + X$ is 4-con'd

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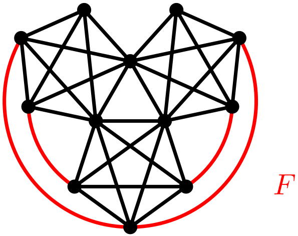

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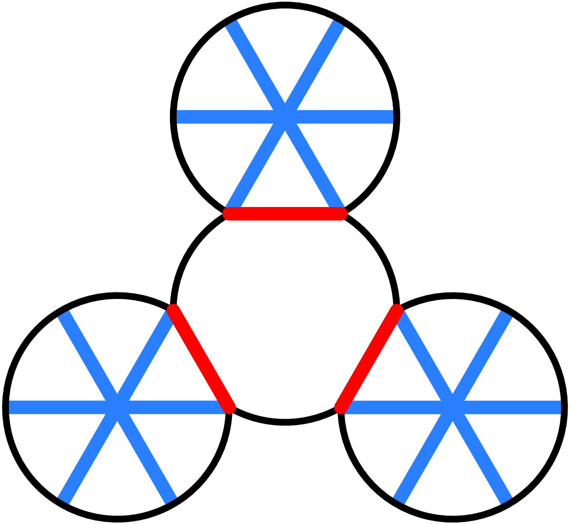
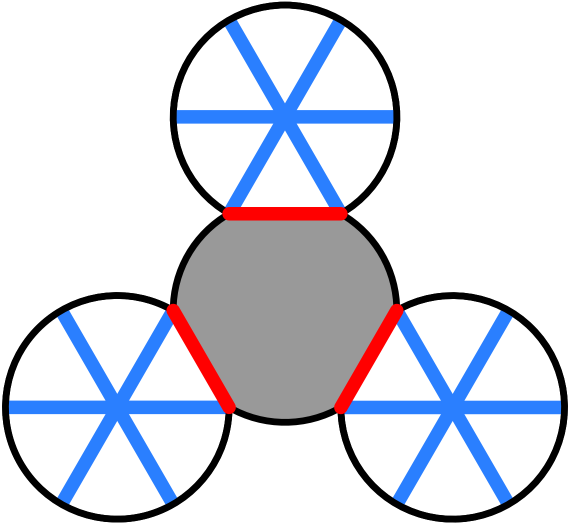
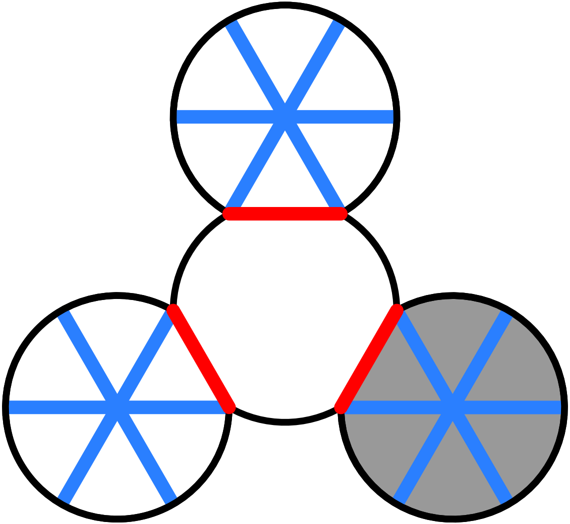
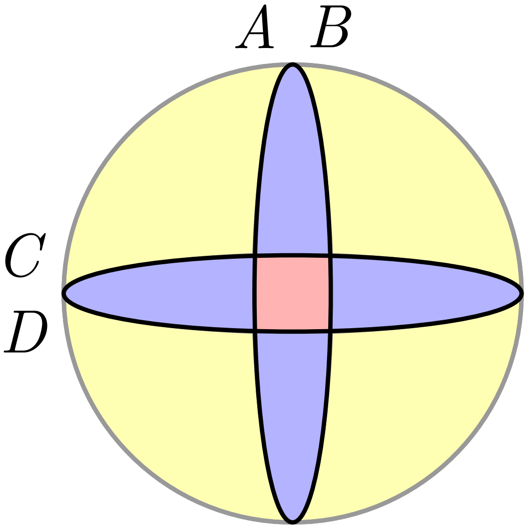
Proof of tetra-decomposition

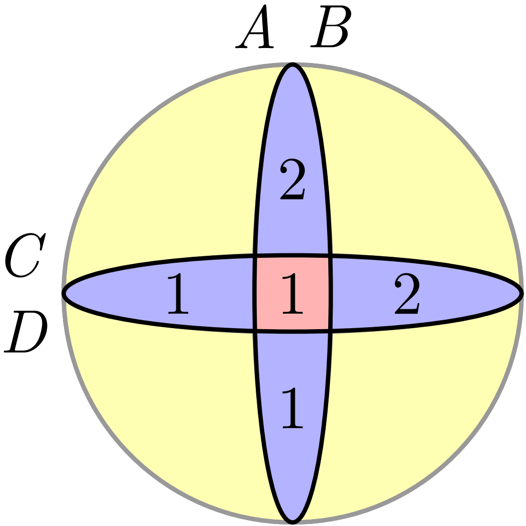




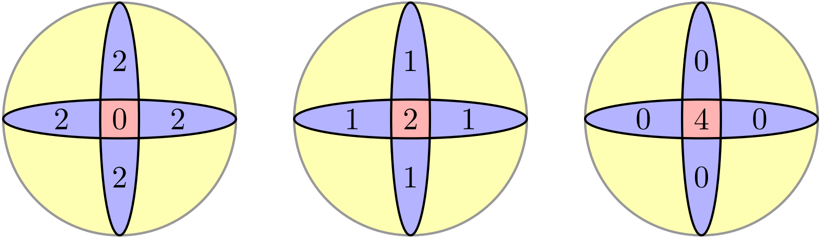


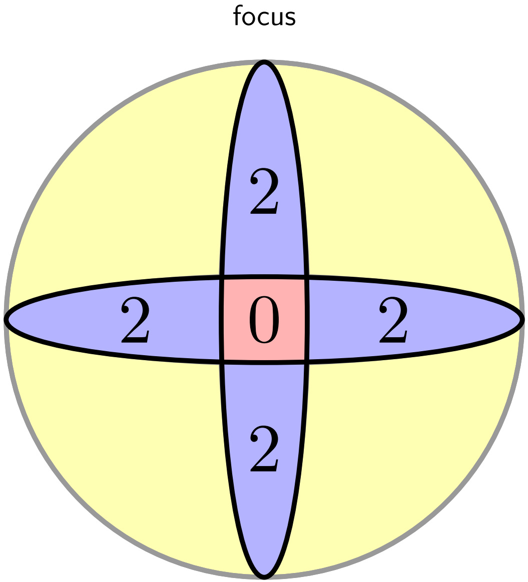


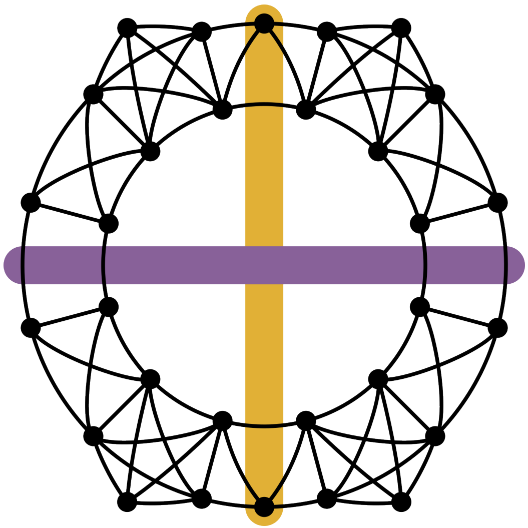
A
B
C
D

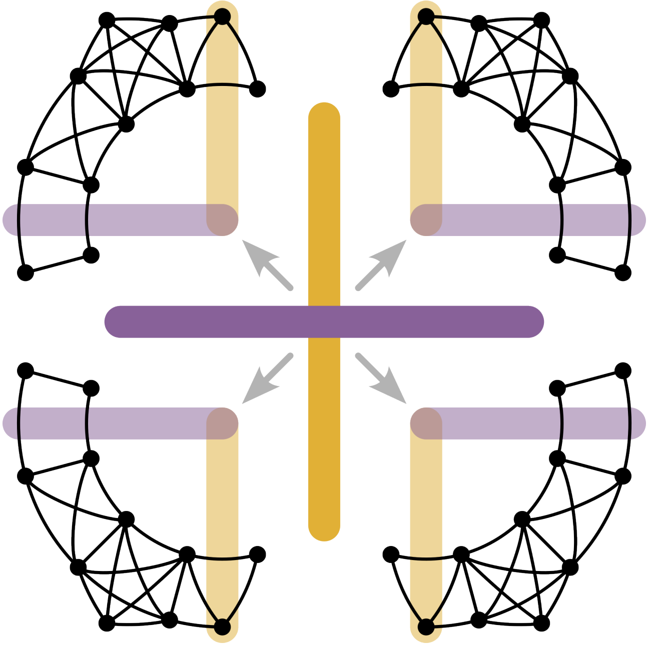

A
B
C
D
2
1
1
2
1

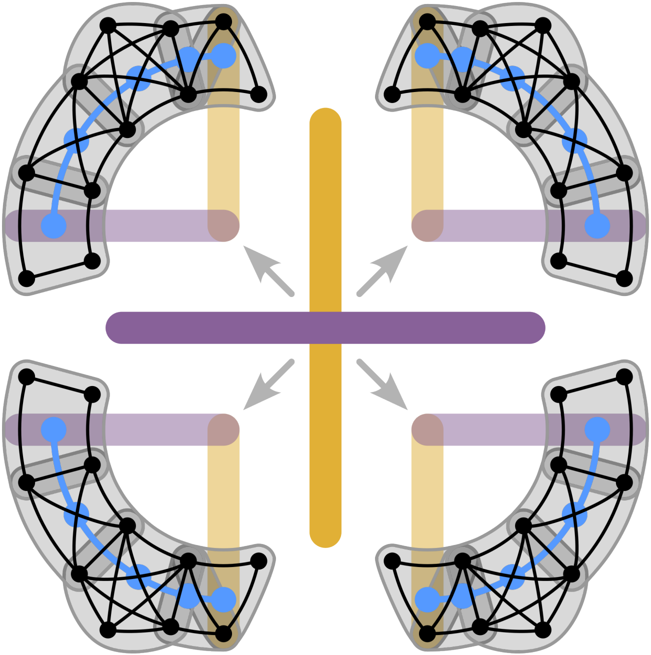
Crossing Lemma. Tetra-sep'ns only cross symmetrically:

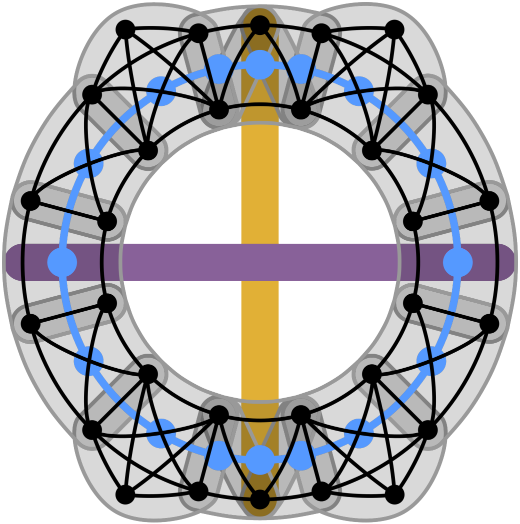


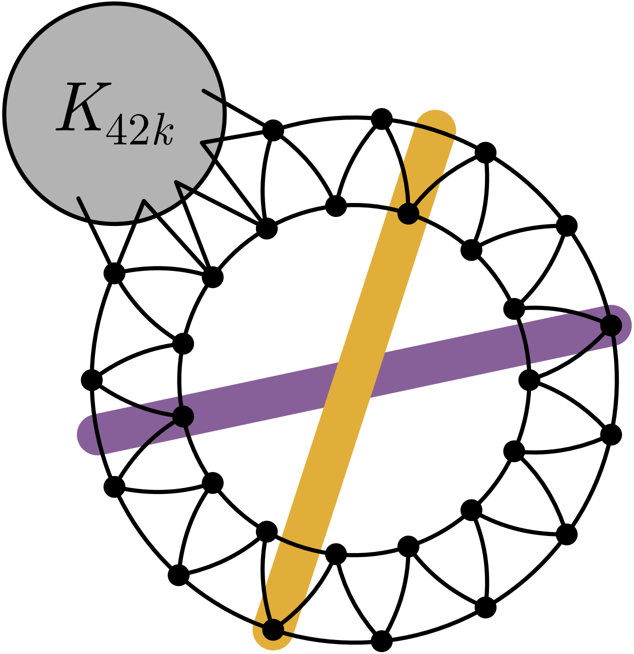

focus
2
2
0
2
2

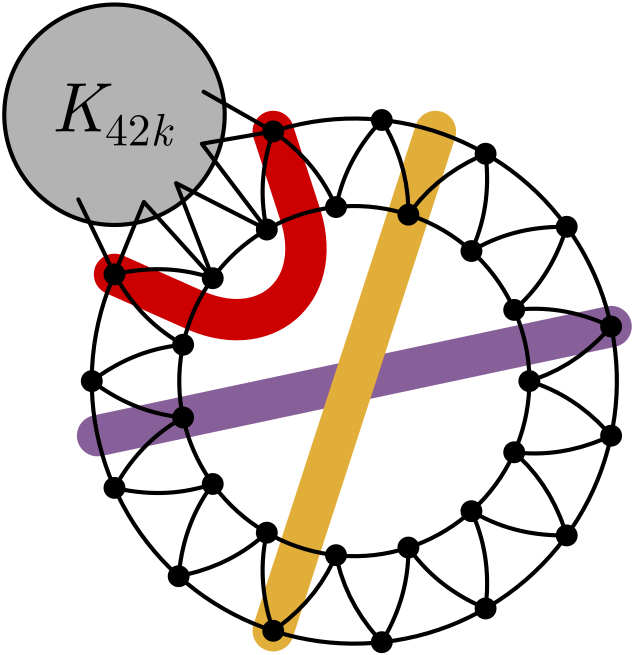






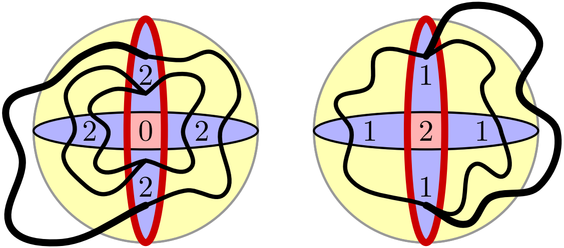


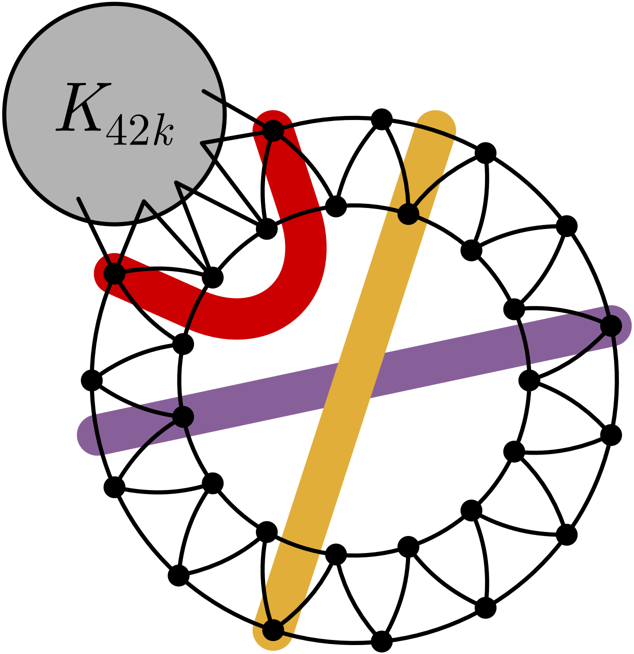

K_{42k}


K_{42k}

(A, B) totally-nested

$\iff$ the sep'r of (A, B) is highly con'd:

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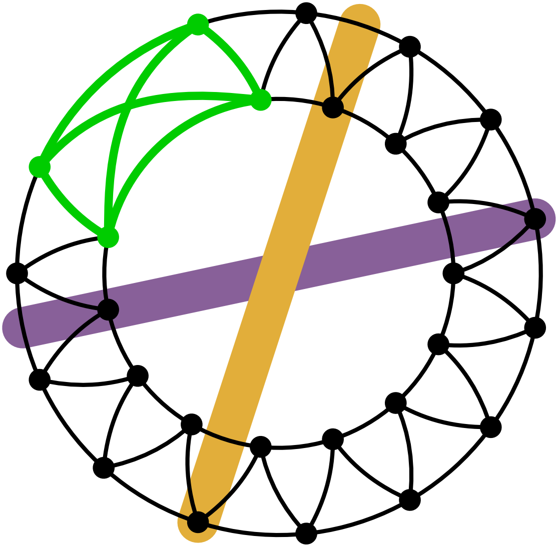

Open: Extend the main result to *all* k.

Open: Directed graphs?

$k = 1$: Bowler, Gut, Hatzel, Kawarabayashi, Muzi, Reich 23

$k \geqslant 2$: ???

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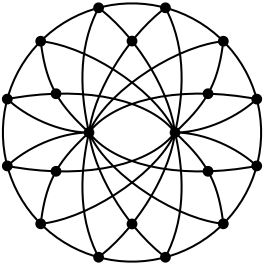
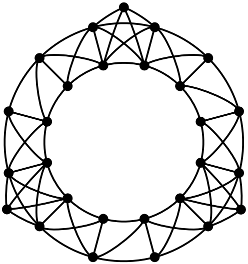


or



Open: Graphs for $k\geqslant 5$. Digraphs for $k\geqslant 2$.

arXiv: 2504.00760 Thank you :) Slides: jan-kurkofka.eu